# CHILEAN JOURNAL OF STATISTICS

Edited by Víctor Leiva

Volume 10 Number 1
April 2019
ISSN: 0718-7912 (print)
ISSN: 0718-7920 (online)

Published by the
Chilean Statistical Society
SOCHE
SOCIEDAD CHILENA DE ESTADÍSTICA

*Chilean Journal of Statistics*
Vol. 10, No. 1, April 2019, 41–53

Goodness-of-fit Methods
Research Paper

# Goodness-of-fit test for the Birnbaum-Saunders distribution based on the Kullback-Leibler information

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(Received: 08 March 2019 · Accepted in final form: 18 April 2019)

**Abstract**

In this work, we propose a goodness-of-fit test based on the Kullback-Leibler information for the Birnbaum-Saunders distribution. We use Monte Carlo simulations to evaluate the size and power of the proposed test for several alternative hypotheses under different sample sizes. We compare the powers with standard goodness-of-fit tests based as the Anderson-Darling and Cramér-von Mises tests. Finally, we illustrate the proposed test with a real data set to show its potential applications.

**Keywords:** Anderson-Darling and Cramér-von Mises tests · Information measures · Maximum likelihood estimation · Monte Carlo method · Power test · R software

**Mathematics Subject Classification:** Primary 62J20 · Secondary 62J99.

## 1. Introduction

The Birnbaum-Saunders (BS) model, proposed by Birnbaum and Saunders (1969), is a life distribution originating from a material fatigue problem, which relates the time to the occurrence of failure with some cumulative damage that is assumed to be Gaussian distributed. The BS model has received much attention in the last decades due to its wide applicability. Based on to its genesis from material fatigue, different cumulative damage processes can be modeled by this distribution, including natural engineering applications, but the BS model can also be applied to other areas as: medicine (Leiva et al., 2007; Barros et al., 2008; Azevedo et al., 2012; Gomes et al., 2012; Desousa et al., 2018; Leao et al., 2018), atmospheric contamination (Leiva et al., 2008, 2010, 2015a; Vilca et al., 2011; Ferreira, 2013; Marchant et al., 2018, 2019), water quality (Leiva et al., 2009; Vilca et al., 2010), neuronal sciences (Leiva et al., 2015b), human aging (Leiva and Saunders, 2015), and earthquakes (Lillo et al., 2018), among others. However, because the BS model is a statistical distribution, we can apply it to several other fields, for example, business, finance, industry, science management, and quality control. For more details about various

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ISSN: 0718-7912 (print)/ISSN 0718-7920 (online)
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http://www.soche.cl/chjs

developments on the BS distribution, see Leiva (2016) and references cited therein. The BS model has also been used to construct new more flexible models having heavier and lighter tails than the standard BS distribution, as well as in the construction of models in the unit interval; see Barros et al. (2008), Azevedo et al. (2012), Mazucheli et al. (2018) and Athayde et al. (2019).

In statistics, it is of great interest to determine whether a probabilistic model fits a data set well or not, which could indicate whether these data may have been generated from this model or not. In this sense, several goodness-of-fit tests have been proposed for different probability distributions. Since goodness-of-fit tests measure the discrepancy between a theoretical model and a data set, they can be done in a variety of ways, such as, for example, formulated by chi-squared type tests, by statistics based on the empirical cumulative distribution function or empirical characteristic function. Further details on goodness-of-fit tests can be found in D'Agostino and Stephens (1986), Castro-Kuriss (2011) and Barros et al. (2014).

The Anderson-Darling (AD) and Cramér-von Mises (CM) statistics are often used to test normality. These statistics are based on the distance between the empirical distribution function and the theoretical distribution function. Chen and Balakrishnam (1995) proposed a general purpose approximate goodness-of-fit test based on these statistics which may be used to test the validity of different families of skew distributions. Note that the Kullback-Leibler (KL) criterion is an information measure, which can be used to evaluate the discrepancy between two distribution functions. Such a measure of information has shown good results in testing fitting of models to data sets, in the sense of obtaining more powerfull tests than the standard tests; see Park (2005) and Rad et al. (2011). Then, due to the wide applicability of the BS distribution, the objective of this paper is to propose a goodness-of-fit test for the BS distribution based on the KL information and investigate if the proposed test is most powerful than in the case of standard AD and CM tests.

The rest of this paper is organized as follows. In Section 2, we present the methodology with the definitions of entropy, KL information, and a brief review of the BS distribution, as well as an estimation method of its parameters. In addition, in this section, goodness-of-fit test for the BS distribution based on KL information are derived. In Section 3, a simulation study based on the Monte Carlo method is conducted to evaluate the size and power of the proposed test. Also in this section, we illustrate the proposed methodology with a real data set. Finally, Section 4 provides the conclusions of this work and some comments on future research related to this topic.

## 2. Methodology

### 2.1 Entropy and Kullback-Leibler information

In order to quantify the degree of disorder in a physical system the German Rudfold Clausius introduced in Clausius (1867) a new quantity in thermodynamics which he called entropy. Since this concept was introduced in studies of information theory by Shannon (1948). Shannon's idea was to measure the degree of disorder of the occurrence of the values of a random variable (RV) in the sense that the more distinct rare events occurr.

Let $X$ be an RV with cumulative distribution function (CDF) $F$ and probability density function (PDF) $f$. The differential entropy $H(f)$ of $X$ is defined in Shannon (1948) by

$$H(f) = -\int_{-\infty}^{\infty} f(x)\log(f(x))\mathrm{d}x.$$

Let $X_1, \ldots, X_n$, with $n \geq 3$, be a sample from the distribution $F$, and let $X_{(1)} \leq \cdots \leq X_{(n)}$

be their corresponding order statistics. A nonparametric estimator of $H(f)$, proposed by Vasicek (1976), is given by

$$H_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{2m}(x_{(i+m)} - x_{(i-m)})\right\}, \tag{1}$$

where the window $m$ is a positive integer less than $n/2$ and $x_{(i-m)} = x_{(1)}$, for $i - m < 1$ and $x_{(i+m)} = x_{(n)}$, for $i+m > n$, such that $x_{(i)}$ is $i$-th observed value of the corresponding order statistic.

Let $f(x)$ and $g(x)$ be PDFs. The KL information is defined in Kullback and Leibler (1951) as

$$I(f\colon g) = \int_{-\infty}^{\infty} f(x)\log\left[\frac{f(x)}{g(x)}\right]\mathrm{d}x, \tag{2}$$

so that $I(f\colon g)$ measures the divergence between the PDFs $f$ and $g$. By using the Gibbs inequality, we can show that $I(f\colon g) \geq 0$ and $I(f\colon g) = 0$ if and only if $f(x) = g(x)$. Thus, the sample estimate of the KL information can also be considered for goodness of fit.

### 2.2 The Birnbaum-Saunders distribution

Let $X$ be a nonnegative RV. Then, $X$ follows a BS distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, if the CDF of $X$ is given by

$$F(x) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}\right)\right], \quad x > 0.$$

We use the notation $X \sim \mathrm{BS}(\alpha, \beta)$ for indicating an RV $X$ with BS distribution of shape and scale parameters $\alpha$ and $\beta$, respectively. Consequently, the PDF of $X$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right]\frac{x^{-3/2}(x+\beta)}{2\alpha\sqrt{\beta}}, \quad x > 0. \tag{3}$$

If $X \sim \mathrm{BS}(\alpha, \beta)$, then the following properties are satisfied:

(i) The parameter $\beta$ is also the median of the distribution.
(ii) If $Z \sim \mathrm{N}(0,1)$, then $X$ and $Z$ are related by $X = \beta(\alpha Z + (\alpha^2 Z^2 + 4)^{1/2})^2/4$. Thus, $Z = (1/\alpha)[(X/\beta)^{1/2} - (\beta/X)^{1/2}] \sim \mathrm{N}(0,1)$.
(iii) $cX \sim \mathrm{BS}(\alpha, c\beta)$, if $c > 0$ and $1/X \sim \mathrm{BS}(\alpha, 1/\beta)$.
(iv) $\mathrm{E}(X) = \beta(1 + \alpha^2/2)$ and $\mathrm{Var}(X) = \beta^2\alpha^2(1 + 5\alpha^2/4)$.
(v) The $q$th quantile of $X$ is given by $x_q = \beta(\alpha z_q + (\alpha^2 z_q^2 + 4)^{1/2})^2/4$, where $z_q = \Phi^{-1}(q)$, $\mathrm{N}(0,1)$ $q$th quantile.
(vi) The survival function is expressed as $S(x;\alpha,\beta) = \Phi\{(1/\alpha)[(\beta/x)^{1/2} - (x/\beta)^{1/2}]\}$.

For estimation of the model parameters, we consider the maximum likelihood (ML) method. Let $X_1, \ldots, X_n$ be a random sample of size $n$ from $X \sim \mathrm{BS}(\alpha, \beta)$ with PDF given by Equation PDF), so that $x_1, \ldots, x_n$ are their respective observed values. Then, the log-likelihood function for $\boldsymbol{\theta} = (\alpha, \beta)^\top$ is given by

$$\ell(\boldsymbol{\theta}) = K - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(\frac{x_i}{\beta} + \frac{\beta}{x_i} - 2\right) + \sum_{i=1}^{n}\log(x_i + \beta) - n\log(\alpha) - \frac{n}{2}\log(\beta),$$

where $K = n(\log(1/\sqrt{2\pi}) - \log(2)) - 3/2\sum_{i=1}^{n}\log(x_i)$. The ML estimate of $\alpha$ is defined as

$$\widehat{\alpha} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\widehat{\beta}} + \frac{\widehat{\beta}}{x_i} - 2\right)}.$$

In the case of the parameter $\beta$, the ML estimate do not have closed form requiring the use of a numerical method. Under regularity conditions (see Cox and Hinkley, 1974), the estimators $\widehat{\alpha}$ and $\widehat{\beta}$ are consistent and have a bivariate normal joint asymptotic distribution with asymptotic means $\alpha$ and $\beta$, respectively, and an asymptotic covariance matrix $\boldsymbol{\Sigma}_{\widehat{\theta}}$ that can obtained from the inverse of the Fisher information matrix given by

$$\mathcal{I}(\boldsymbol{\theta}) = \begin{pmatrix} \frac{2n}{\alpha^2} & 0 \\ 0 & \frac{n}{\beta^2}(\frac{1}{4} + \frac{1}{\alpha^2} + I(\alpha)) \end{pmatrix},$$

where

$$I(\alpha) = 2\int_0^\infty \left(\frac{1}{1 + \frac{1}{\xi(az)}} - \frac{1}{2}\right)^2 \phi(z)\mathrm{d}z,$$

with $\phi$ being the PDF of $Z \sim \mathrm{N}(0, 1)$ and $\xi(u) = u^{1/2} - u^{-1/2}$. For more details, see Leiva (2016).

## 2.3 Goodness-of-fit tests for the BS distribution

Given a random sample $X_1, \ldots, X_n$ of the RV $X$, we are interested in testing $\mathrm{H}_0$: the RV $X$ follows the BS($\alpha$, $\beta$) distribution with PDF given in Equation (3) against $\mathrm{H}_1$: the RV $X$ does not follow the BS distribution. Note that Equation (2) can be written as

$$\begin{aligned} I(f\colon g) &= \int_{-\infty}^{\infty} f(x)[\log(f(x)) - \log(g(x))]\mathrm{d}x \\ &= -H(f) - \int_{-\infty}^{\infty} f(x)\log(g(x))\mathrm{d}x. \end{aligned} \quad (4)$$

Then, from Equation (4), an estimate of the KL information can be obtained. For doing this, we replace $H(f)$ by its estimate given in Equation (1) and we use the estimated values of the parameters in $f$. Thus, under the null hypothesis that $f(x) = g(x)$, we can estimate the information of KL using

$$I_{mn} = -H_{mn} - \int_{-\infty}^{\infty} f(x; \widehat{\boldsymbol{\theta}})\log(f(x; \widehat{\boldsymbol{\theta}}))\mathrm{d}x,$$

where $\widehat{\boldsymbol{\theta}}$ is a consistent estimator for $\boldsymbol{\theta}$. Therefore, $I_{mn}$ is a test statistic to verify the suitability of a continuous probabilistic model with PDF given by $f$ to a data set.

For $X \sim \mathrm{BS}(\alpha, \beta)$ and $f$ given in Equation (3), we obtain

$$I_{mn} = -H_{mn} - \log \frac{1}{\sqrt{2\pi}} - \frac{1}{\widehat{\alpha}^2} + \log\left(2\widehat{\alpha}\sqrt{\widehat{\beta}}\right) + \frac{1}{\widehat{\alpha}^2}\left(1 + \frac{\widehat{\alpha}^2}{2}\right)$$
$$+ \frac{3}{2n}\sum_{i=1}^{n}\log(x_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}\log(x_{(i)} + \widehat{\beta}),$$

where $\widehat{\alpha}$ and $\widehat{\beta}$ are the ML estimates of $\alpha$ and $\beta$, respectively. Thus, following Arizono and Ohta (1989), we introduce the statistic

$$\mathrm{KL}_{mn} = \frac{1}{\exp(I_{mn})},$$

with $0 \leq \mathrm{KL}_{mn} \leq 1$ since $I_{mn} \in [0, \infty)$. Note that $\mathrm{KL}_{mn}$ can be used as test statistic for testing the goodness-of-fit of the BS distribuion to a data set. The decision rule is to reject the hypothesis $\mathrm{H}_0$ if $\mathrm{KL}_{mn} \leq \mathrm{KL}^*_{mn}(\rho)$, where $\mathrm{KL}^*_{mn}(\rho)$ is the critical value for a significance level $\rho$. As we do not have an exact distribution of $\mathrm{KL}_{mn}$, then we obtain $\mathrm{KL}^*_{mn}(\rho)$ through Monte Carlo simulations.

## 3. Numerical Studies

### 3.1 Critical values for the simulations

To obtain the critical values of the proposed test, we conduct Monte Carlo simulation studies with $R = 10,000$ replications each. These studies are based on $n \in \{10, 30, 50, 100\}$, $\alpha \in \{0.5, 1.0, 1.5\}$, and significance level $\rho = 0.05$. In addition, we fix, without loss of generality, $\beta = 1$, since this is a scale parameter. The values considered for the window $m$ are those returned the maximum critical value, according to Arizono and Ohta (1989). This procedure is described in Algorithm 1. All simulations are obtained from implementations in the `R` statistical software, which is freely distributed from `www.R-project.org`. For parameters estimation we use the `maxLik` package.

**Algorithm 1:** Obtaining the critical values of the proposed test.

1: Fix $n$, $\alpha$ and $\beta$;
2: Generate 10,000 random samples of size $n$ from $X \sim \mathrm{BS}(\alpha, \beta)$;
3: For each sample, estimate the parameter vector $\boldsymbol{\theta} = (\alpha, \beta)^\top$ consistently, through the ML method;
4: For each sample, obtain the values of the test statistic $\mathrm{KL}_{mn}$;
5: Sort the test statistic values obtained in the previous step and determine the 5th quantile and then obtain the critical values for the respective significance level.

The critical values obtained, considering the BS(0.5,1), BS(1,1) and BS(1.5,1) distributions are presented in Tables 1-3.

### 3.2 Evaluating the empirical size and power of the test

Next, the empirical size and power of the proposed test are evaluated for different sample sizes based on the Monte Carlo method. We make a comparison among the AD, CM and KL tests, whose statistics are denoted by $A^2, W^2$, KL, and verify in what situations the test based on the KL information is better, in the sense of being most powerful.

Table 1. Critical values for the statistic $\mathrm{KL}_{mn}$ considering the BS(0.5,1) distribution and significance level 5%.

| | $m$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 0.2462 | | | | | | | | | |
| 4 | 0.2577 | | | | | | | | | |
| 5 | 0.2925 | 0.4221 | | | | | | | | |
| 6 | 0.3256 | 0.4404 | | | | | | | | |
| 7 | 0.3544 | 0.4620 | 0.4835 | | | | | | | |
| 8 | 0.3866 | 0.4935 | 0.5083 | | | | | | | |
| 9 | 0.4054 | 0.5102 | 0.5319 | 0.5168 | | | | | | |
| 10 | 0.4250 | 0.5340 | 0.5481 | 0.5401 | | | | | | |
| 12 | 0.4614 | 0.5689 | 0.5840 | 0.5760 | 0.5625 | | | | | |
| 14 | 0.4911 | 0.5908 | 0.6114 | 0.6072 | 0.5973 | 0.5771 | | | | |
| 16 | 0.5159 | 0.6207 | 0.6383 | 0.6354 | 0.6227 | 0.6069 | 0.5880 | | | |
| 18 | 0.5308 | 0.6396 | 0.6597 | 0.6605 | 0.6461 | 0.6331 | 0.6184 | 0.5980 | | |
| 20 | 0.5499 | 0.6564 | 0.6820 | 0.6796 | 0.6674 | 0.6542 | 0.6428 | 0.6250 | 0.6082 | |
| 25 | 0.5754 | 0.6871 | 0.7176 | 0.7194 | 0.7124 | 0.7042 | 0.6905 | 0.6769 | 0.6617 | 0.6489 |
| 30 | 0.5976 | 0.7132 | 0.7421 | 0.7474 | 0.7481 | 0.7384 | 0.7280 | 0.7153 | 0.7036 | 0.6899 |
| 35 | 0.6122 | 0.7297 | 0.7593 | 0.7699 | 0.7707 | 0.7655 | 0.7577 | 0.7473 | 0.7352 | 0.7254 |
| 40 | 0.6243 | 0.7423 | 0.7766 | 0.7904 | 0.7900 | 0.7860 | 0.7789 | 0.7720 | 0.7620 | 0.7527 |
| 45 | 0.6343 | 0.7547 | 0.7887 | 0.8007 | 0.8053 | 0.8034 | 0.7975 | 0.7917 | 0.7832 | 0.7765 |
| 50 | 0.6426 | 0.7634 | 0.7982 | 0.8129 | 0.8165 | 0.8142 | 0.8146 | 0.8062 | 0.8027 | 0.7935 |
| 60 | 0.6568 | 0.7755 | 0.8135 | 0.8291 | 0.8350 | 0.8368 | 0.8355 | 0.8330 | 0.8274 | 0.8235 |
| 70 | 0.6646 | 0.7854 | 0.8251 | 0.8421 | 0.8498 | 0.8515 | 0.8522 | 0.8501 | 0.8476 | 0.8435 |
| 80 | 0.6751 | 0.7959 | 0.8349 | 0.8514 | 0.8596 | 0.8641 | 0.8644 | 0.8649 | 0.8628 | 0.8595 |
| 90 | 0.6804 | 0.8012 | 0.8408 | 0.8598 | 0.8687 | 0.8735 | 0.8758 | 0.8742 | 0.8733 | 0.8718 |
| 100 | 0.6858 | 0.8075 | 0.8471 | 0.8656 | 0.8760 | 0.8818 | 0.8833 | 0.8841 | 0.8826 | 0.8813 |

Under same the conditions of the obtained critical values, we calculate the empirical size of the test. Algorithm 2 displays this procedure. The results of our simulation study are presented in Table 4. Note that the empirical size is close to the nominal level for all situations considered, indicating that the test is controlled.

**Algorithm 2:** Obtaining the empirical size of the proposed test.

1: Fix $n$, $\alpha$ and $\beta$;
2: Generate 10,000 random samples of size $n$ from $X \sim \mathrm{BS}(\alpha, \beta)$;
3: For each sample, estimate the parameter vector $\boldsymbol{\theta} = (\alpha, \beta)^\top$ consistently, through the ML method;
4: For each sample, obtain the values of the test statistic $\mathrm{KL}_{mn}$;
5: Obtain the empirical size of the test by calculating the proportion of replications that present test statistic value less than the critical value for the corresponding values of $n$ and $m$.

To determine the empirical power, we consider some probability distributions for the alternative hypothesis. These distributions are chosen and grouped into classes to be analyzed according to the shape of their hazard function: increasing, decreasing and non-monotonous. The probability distributions considered in the evaluation of the power test

Table 2. Critical values for the statistic $\mathrm{KL}_{mn}$ considering the BS(1,1) distribution and significance level 5%.

| $n$ | $m$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.2618 | | | | | | | | | |
| 4 | 0.2724 | | | | | | | | | |
| 5 | 0.3066 | 0.4446 | | | | | | | | |
| 6 | 0.3369 | 0.4698 | | | | | | | | |
| 7 | 0.3653 | 0.4947 | 0.5095 | | | | | | | |
| 8 | 0.3974 | 0.5215 | 0.5398 | | | | | | | |
| 9 | 0.4132 | 0.5349 | 0.5650 | 0.5420 | | | | | | |
| 10 | 0.4350 | 0.5575 | 0.5809 | 0.5691 | | | | | | |
| 12 | 0.4681 | 0.5870 | 0.6143 | 0.6133 | 0.5941 | | | | | |
| 14 | 0.4960 | 0.6066 | 0.6390 | 0.6408 | 0.6338 | 0.6115 | | | | |
| 16 | 0.5202 | 0.6338 | 0.6610 | 0.6654 | 0.6591 | 0.6464 | 0.6292 | | | |
| 18 | 0.5342 | 0.6508 | 0.6803 | 0.6871 | 0.6807 | 0.6709 | 0.6592 | 0.6416 | | |
| 20 | 0.5539 | 0.6671 | 0.6974 | 0.7049 | 0.6989 | 0.6923 | 0.6854 | 0.6683 | 0.6558 | |
| 25 | 0.5778 | 0.6946 | 0.7299 | 0.7387 | 0.7371 | 0.7358 | 0.7279 | 0.7177 | 0.7056 | 0.6986 |
| 30 | 0.5987 | 0.7193 | 0.7525 | 0.7621 | 0.7686 | 0.7639 | 0.7587 | 0.7502 | 0.7436 | 0.7357 |
| 35 | 0.6140 | 0.7341 | 0.7680 | 0.7823 | 0.7863 | 0.7856 | 0.7827 | 0.7776 | 0.7709 | 0.7651 |
| 40 | 0.6258 | 0.7460 | 0.7830 | 0.7990 | 0.8038 | 0.8031 | 0.7997 | 0.7968 | 0.7911 | 0.7869 |
| 45 | 0.6355 | 0.7572 | 0.7942 | 0.8089 | 0.8156 | 0.8169 | 0.8157 | 0.8122 | 0.8085 | 0.8053 |
| 50 | 0.6436 | 0.7663 | 0.8028 | 0.8196 | 0.8261 | 0.8261 | 0.8289 | 0.8249 | 0.8231 | 0.8175 |
| 60 | 0.6575 | 0.7776 | 0.8169 | 0.8338 | 0.8422 | 0.8453 | 0.8471 | 0.8463 | 0.8431 | 0.8427 |
| 70 | 0.6651 | 0.7872 | 0.8279 | 0.8461 | 0.8555 | 0.8583 | 0.8603 | 0.8606 | 0.8596 | 0.8577 |
| 80 | 0.6753 | 0.7967 | 0.8375 | 0.8546 | 0.8637 | 0.8694 | 0.8717 | 0.8730 | 0.8727 | 0.8716 |
| 90 | 0.6806 | 0.8024 | 0.8429 | 0.8621 | 0.8722 | 0.8781 | 0.8815 | 0.8809 | 0.8816 | 0.8809 |
| 100 | 0.6859 | 0.8084 | 0.8484 | 0.8675 | 0.8789 | 0.8850 | 0.8877 | 0.8898 | 0.8895 | 0.8895 |

are: gamma, generalized exponential, beta, Pareto type I, Weibull, and half-normal, whose PDFs are the following:

- Gamma($\kappa;\theta$) with PDF

$$f_1(x;\kappa,\theta) = \frac{1}{\Gamma(\kappa)\theta^\kappa} x^{\kappa-1} \exp\left(-\frac{x}{\theta}\right), \quad x > 0,\ \kappa,\theta > 0$$

  and CDF denoted by $F_1$.
- GExp($\kappa;\theta$) with PDF

$$f_2(x;\kappa,\theta) = \kappa\theta x \exp\{-\theta x\}[1-\exp(-\theta x)]^{\kappa-1}, \quad x > 0,$$

  $\kappa,\theta > 0$, and CDF denoted by $F_2$.
- Beta($\kappa;\theta$), with PDF

$$f_3(x;\kappa,\theta) = \frac{\Gamma(\kappa+\theta)}{\Gamma(\kappa)\Gamma(\theta)} x^{\kappa-1}(1-x)^{\theta-1}, \quad 0 < x < 1,$$

  $\kappa,\theta > 0$, and CDF denoted by $F_3$.

Table 3. Critical values for the statistic $\mathrm{KL}_{mn}$ considering the BS(1.5,1) distribution and significance level 5%.

| | $m$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 0.2819 | | | | | | | | | |
| 4 | 0.2911 | | | | | | | | | |
| 5 | 0.3237 | 0.4760 | | | | | | | | |
| 6 | 0.3514 | 0.5053 | | | | | | | | |
| 7 | 0.3796 | 0.5303 | 0.5440 | | | | | | | |
| 8 | 0.4102 | 0.5547 | 0.5791 | | | | | | | |
| 9 | 0.4256 | 0.5665 | 0.6065 | 0.5852 | | | | | | |
| 10 | 0.4449 | 0.5865 | 0.6206 | 0.6114 | | | | | | |
| 12 | 0.4763 | 0.6121 | 0.6529 | 0.6591 | 0.6446 | | | | | |
| 14 | 0.5036 | 0.6274 | 0.6734 | 0.6850 | 0.6848 | 0.6684 | | | | |
| 16 | 0.5270 | 0.6509 | 0.6916 | 0.7056 | 0.7080 | 0.7029 | 0.6938 | | | |
| 18 | 0.5385 | 0.6651 | 0.7067 | 0.7234 | 0.7267 | 0.7247 | 0.7198 | 0.7136 | | |
| 20 | 0.5564 | 0.6787 | 0.7191 | 0.7371 | 0.7430 | 0.7454 | 0.7462 | 0.7343 | 0.7317 | |
| 25 | 0.5806 | 0.7025 | 0.7460 | 0.7630 | 0.7704 | 0.7791 | 0.7791 | 0.7785 | 0.7741 | 0.7770 |
| 30 | 0.6010 | 0.7260 | 0.7643 | 0.7816 | 0.7940 | 0.7992 | 0.8004 | 0.8024 | 0.8021 | 0.8048 |
| 35 | 0.6149 | 0.7396 | 0.7777 | 0.7984 | 0.8080 | 0.8131 | 0.8167 | 0.8198 | 0.8209 | 0.8238 |
| 40 | 0.6273 | 0.7499 | 0.7900 | 0.8116 | 0.8210 | 0.8254 | 0.8293 | 0.8318 | 0.8340 | 0.8349 |
| 45 | 0.6365 | 0.7602 | 0.8008 | 0.8193 | 0.8298 | 0.8358 | 0.8404 | 0.8425 | 0.8447 | 0.8478 |
| 50 | 0.6443 | 0.7685 | 0.8080 | 0.8285 | 0.8385 | 0.8425 | 0.8496 | 0.8507 | 0.8541 | 0.8544 |
| 60 | 0.6581 | 0.7794 | 0.8206 | 0.8402 | 0.8507 | 0.8572 | 0.8614 | 0.8645 | 0.8661 | 0.8708 |
| 70 | 0.6657 | 0.7884 | 0.8309 | 0.8506 | 0.8618 | 0.8677 | 0.8720 | 0.8749 | 0.8778 | 0.8789 |
| 80 | 0.6758 | 0.7977 | 0.8397 | 0.8580 | 0.8694 | 0.8761 | 0.8808 | 0.8847 | 0.8867 | 0.8886 |
| 90 | 0.6807 | 0.8026 | 0.8449 | 0.8652 | 0.8760 | 0.8838 | 0.8891 | 0.8904 | 0.8934 | 0.8946 |
| 100 | 0.6856 | 0.8089 | 0.8502 | 0.8699 | 0.8824 | 0.8895 | 0.8936 | 0.8972 | 0.8988 | 0.9012 |

Table 4. Empirical size for different sample size and values of the parameter $\alpha$ indicated.

| $n$ | $m$ | BS(0.5,1) | BS(1,1) | BS(1.5,1) |
|---|---|---|---|---|
| 10 | 3 | 0.0473 | 0.0588 | 0.0494 |
| 30 | 5 | 0.0564 | 0.0563 | 0.0513 |
| 50 | 7 | 0.0520 | 0.0514 | 0.0454 |
| 100 | 8 | 0.0504 | 0.0515 | 0.0482 |

- Pareto($\kappa;\theta$), with PDF

$$f_4(x;\kappa,\theta) = \frac{\kappa\theta^\kappa}{x^{\kappa+1}}, \quad x \in [\theta,\infty),\ \kappa,\theta > 0,$$

  and CDF denoted by $F_4$.
- Weibull($\kappa;\theta$), with PDF

$$f_5(x;\kappa,\theta) = \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\exp\left\{-\left(\frac{x}{\theta}\right)^{\kappa}\right\}, \quad x > 0,$$

  $\kappa,\theta > 0$ and CDF denoted by $F_5$.

- HN($\theta$), with PDF

$$f_6(x;\theta) = \frac{2\theta}{\pi}\exp\left(-\frac{x^2\theta^2}{\pi}\right), \quad x \geq 0,\ \theta > 0,$$

and CDF denoted by $F_6$.

The power of the test is calculated based on testing the hypotheses

$$\begin{cases} \mathrm{H}_0\colon\ X \sim \mathrm{BS}(\alpha,\beta),\ \text{for some } \alpha > 0 \text{ and } \beta > 0; \\ \mathrm{H}_1\colon\ X \sim F_i(\boldsymbol{\theta}),\ \text{with } \boldsymbol{\theta} > 0 \text{ and } i = 1,\ldots,6. \end{cases}$$

In the procedure, 10,000 Monte Carlo replications and sample sizes $n = 10, 30, 50, 100$ are considered. The powers of the tests are obtained at the significance level $\rho = 0.05$. For each value of $n$ and each distribution in $\mathrm{H}_1$, with different parameters, the 10,000 samples are generated and the respective values of the test statistic are calculated. Based on the critical values presented in Tables 1-3, we obtain the rejection proportions based on the 10,000 simulated samples. In addition, the power of the test is evaluated based on the CM and AD statistics using the procedure proposed by Chen and Balakrishnam (1995). We make a comparison among the tests and verify in what situations the test based on the KL information is better, in the sense of being most powerful. Tables 5-8 present the powers for the test in question with sample sizes of $n = 10, n = 30$, $n = 50$ and $n = 100$, respectively.

Table 5. Empirical power for different forms of hazard functions and different distributions considering sample size $n = 10$.

| Hazard function | Alternatives | $\mathrm{KL}_{mn}$ | $W^2$ | $A^2$ |
|---|---|---|---|---|
| Increasing | Gamma(3; 1) | 0.1534 | 0.0873 | 0.0937 |
| | GExp(3; 1) | 0.1288 | 0.0805 | 0.0825 |
| | Beta(2; 1) | 0.6841 | 0.3970 | 0.4282 |
| Decreasing | Gamma(0.5; 1) | 0.0376 | 0.0890 | 0.0959 |
| | GExp(0.5; 1) | 0.0428 | 0.0938 | 0.1025 |
| Nonmonotone | Pareto(2; 1) | 0.4748 | 0.4070 | 0.4342 |
| | Weibull(2; 1) | 0.2405 | 0.1507 | 0.1617 |
| | HN(3) | 0.2434 | 0.2096 | 0.2298 |

Table 6. Empirical power for different forms of hazard functions and different distributions considering sample size $n = 30$.

| Hazard function | Alternatives | $\mathrm{KL}_{mn}$ | $W^2$ | $A^2$ |
|---|---|---|---|---|
| Increasing | Gamma(3; 1) | 0.2656 | 0.1856 | 0.2082 |
| | GExp(3; 1) | 0.2050 | 0.1544 | 0.1719 |
| | Beta(2; 1) | 0.9970 | 0.9053 | 0.9388 |
| Decreasing | Gamma(0.5; 1) | 0.3465 | 0.3959 | 0.5343 |
| | GExp(0.5; 1) | 0.3638 | 0.3937 | 0.5442 |
| Nonmonotone | Pareto(2; 1) | 0.9767 | 0.9039 | 0.9365 |
| | Weibull(2; 1) | 0.5458 | 0.4172 | 0.4559 |
| | HN(3) | 0.7164 | 0.6576 | 0.6987 |

Table 7. Empirical power for different forms of hazard functions and different distributions considering sample size $n = 50$.

| Hazard function | Alternatives | $\mathrm{KL}_{mn}$ | $W^2$ | $A^2$ |
|---|---|---|---|---|
| Increasing | Gamma(3; 1) | 0.3575 | 0.2774 | 0.3099 |
| | GExp(3; 1) | 0.2716 | 0.2187 | 0.2498 |
| | Beta(2; 1) | 1.0000 | 0.9905 | 0.9965 |
| Decreasing | Gamma(0.5; 1) | 0.6622 | 0.7369 | 0.8711 |
| | GExp(0.5; 1) | 0.6779 | 0.7394 | 0.8748 |
| Nonmonotone | Pareto(2; 1) | 0.9993 | 0.9922 | 0.9970 |
| | Weibull(2; 1) | 0.7317 | 0.6190 | 0.6671 |
| | HN(3) | 0.9026 | 0.8701 | 0.8999 |

Table 8. Empirical power for different forms of hazard functions and different distributions considering sample size $n = 100$.

| Hazard function | Alternatives | $\mathrm{KL}_{mn}$ | $W^2$ | $A^2$ |
|---|---|---|---|---|
| Increasing | Gamma(3; 1) | 0.4937 | 0.4829 | 0.5364 |
| | GExp(3; 1) | 0.3744 | 0.3807 | 0.4276 |
| | Beta(2; 1) | 1.0000 | 1.0000 | 1.0000 |
| Decreasing | Gamma(0.5; 1) | 0.9861 | 0.9786 | 0.9969 |
| | GExp(0.5; 1) | 0.9882 | 0.9772 | 0.9957 |
| Nonmonotone | Pareto(2; 1) | 1.0000 | 1.0000 | 1.0000 |
| | Weibull(2; 1) | 0.9134 | 0.8922 | 0.9256 |
| | HN(3) | 0.9948 | 0.9915 | 0.9958 |

According to our simulation study, we conclude that the goodness-of-fit test based on the KL information, in general, presents greater powers when compared to standard AD and CM tests, for small sample size. When the hazard function under alternative hypothesis is decreasing, the proposed test has difficulties in discriminating the models, leading to powers close to nominal levels. This is because the hazard functions considered under the alternative hypothesis closely approximate the hazard function of the BS distribution. In addition, as the sample size increases, the power of the test also increases, as expected.

### 3.3 Empirical illustration

Next, we consider a set of data related to fatigue life cycles of samples of 6061-T6 aluminum presented in Birnbaum and Saunders (1969). These specimens were cut at an angle parallel to the direction of rotation, oscillating at 18 cycles per second. They were exposed to a pressure with a maximum stress of 26000 psi (pounds per square inch). The data are presented in Table 9.

We want to test the null hypothesis that the sample presented in Table 9 follows the BS distribution.The model parameter estimates are $\widehat{\alpha} = 0.1614$ and $\widehat{\beta} = 392.7622$. The value observed for the test statistic is $\mathrm{kl}_{mn} = 0.9270$, and the critical value for this case is $\mathrm{KL}^*_{mn}(\rho) = 0.8834$, at the 5% significance level. Therefore, we do not reject the hypothesis that the data follow the BS distribution. Figure 1 compares the empirical distribution function with the theoretical one. We can observe from this figure that the empirical and theoretical distribution functions are very close, which reinforces the conclusion reached by the test.

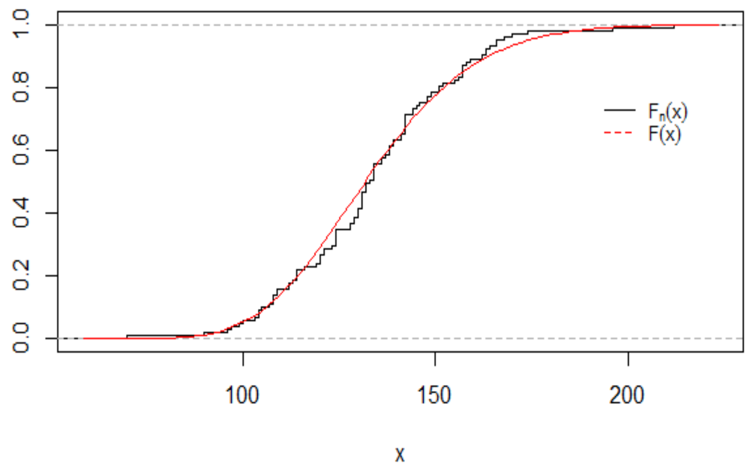


Figure 1. Empirical and theoretical distribution functions BS for aluminum data.

Table 9. Data set of aluminum lifetimes (26.000 psi).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 233 | 258 | 268 | 276 | 290 | 310 | 312 | 315 | 318 | 321 |
| 321 | 329 | 335 | 336 | 338 | 338 | 342 | 342 | 342 | 344 |
| 349 | 350 | 350 | 351 | 351 | 352 | 352 | 356 | 358 | 358 |
| 360 | 362 | 363 | 366 | 367 | 370 | 370 | 372 | 372 | 374 |
| 375 | 376 | 379 | 379 | 380 | 382 | 389 | 389 | 395 | 396 |
| 400 | 400 | 400 | 403 | 404 | 406 | 408 | 408 | 410 | 412 |
| 414 | 416 | 416 | 416 | 420 | 422 | 423 | 426 | 428 | 432 |
| 432 | 433 | 433 | 437 | 438 | 439 | 439 | 443 | 445 | 445 |
| 452 | 456 | 456 | 460 | 464 | 466 | 468 | 470 | 470 | 473 |
| 474 | 476 | 476 | 486 | 488 | 489 | 490 | 491 | 503 | 517 |
| 540 | 560 | | | | | | | | |

## 4. Conclusions and Future Research

In this paper, we proposed a goodness-of-fit test for the Birnbaum-Saunders distribution based on the Kullback-Leibler information. The proposed goodness-of-fit test performed better than the standard Anderson-Darling and Cramér-von Mises tests, in the sense that the proposed test had greater power for the alternatives considered with increasing and nonmonotone hazard functions. When the distribution of the alternative hypothesis had a decreasing hazard function, the test based in KL information presented less power than the Anderson-Darling and Cramér-von Mises tests. In general, the proposed test proved to be a good alternative to the standard Anderson-Darling and Cramér-von Mises tests. As future research, we hope to obtain new tests for the Birnbaum-Saunders distribution based on information measures for censored data, more specifically, for type II and progressively Type-II censored samples.

## Acknowledgment

The authors wish to thank the Editors and anonymous referees for their constructive comments on an earlier version of this manuscript, which resulted in this improved version. The study was supported partially by CNPq and CAPES.

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