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Bayesian computational methods for estimation of two-parameters Weibull distribution in presence of right-censored data

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Abstract

In this paper we study the performance of the Metropolis-Hastings and Slice sampling algorithms for estimating the Weibull distribution parameters. The Metropolis-Hastings algorithm is developed considering its two main version namely independent Metropolis-Hastings and random walk Metropolis. A numerical simulation study is carried out to understand performance of the three methods and compare their performances with maximum likelihood estimation. The comparison among methods is made in terms of sample root mean square errors and bias. We find that the Random walk Metropolis and Slice sampling present a complementary behaviour and outperforms the Independent Metropolis-Hastings and Maximum Likelihood estimation, specially, for datasets with censored times. The methods are also illustrated on three real datasets.

Keywords: Weibull distribution · Bayesian inference · Metropolis-Hastings · Slice sampling.

Mathematics Subject Classification: Primary 62F15 · Secondary 62F40.

1. Introduction

Since its introduction in 1951 by professor Wallodi Weibull (Weibull, 1951), the Weibull distribution has been successfully used to model a very extensive variety to complex mechanisms, such that, reliability engineering (Mann et al., 1974; Johson et al., 1994), lifetime data (Lawless, 1974; Kleinbaum and Klein, 2012; Collet, 2003), among others. It is main due its versatility, in which, depending on the parameters values an increasing, constant or decreasing hazard rate can be modelled. For more details, please see the books of Rinne (2008) and McColl (1998).

In the literature there exist several versions of the Weibull distribution, since with two or three parameters as well as modifications such as Inverse Weibull distribution, Truncated Weibull distribution, mixed Weibull distributions, compound Weibull distributions and extended Weibull distributions with four and five parameters (Rinne, 2008). This paper focuses on the two-parameter version.

Due its broad applicability, methods to estimate Weibull model parameters precisely and efficiently are very important. The two most used estimation methods are the method of

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maximum likelihood (ML) and method of moments (MM). But, there are various different estimation methods in the literature for estimating the parameters of the Weibull distribution. Among them, last square method and weighted least square method (Pobočíková and Sedliacková, 2014; Nwobi and Ugomma, 2014), L-moments method (Hosking, 1990), probability weighted moment estimator (Bartoluccia et al., 1999), percentile estimators (Dubey, 1967) and methods based on modification of MM and ML (Cohen, 1982; McColl, 1998; Ahmad, 1994). But, especially due to their good properties and for being easy to be calculated, ML remains the most popular estimation method for estimating the Weibull distribution parameters.

In this paper, we develop a Bayesian approach to estimate the Weibull distribution parameters under random right-censored data. The two unknown parameters of Weibull distribution are denoted by α and β. The estimation procedure is carried out via Markov chain Monte Carlo (MCMC). The scale parameter β is updated via Gibbs sampling. To update shape parameter α we consider the Metropolis-Hastings algorithm (Chib and Greenberg, 1995) in its two main versions: Independent Metropolis-Hastings (IMH) and Random Walk Metropolis (RWM). But, as in some cases, creating a good candidate generating-density in MH may be difficult, so we also describe as to update shape parameter α via Slice sampling (Neal, 2003).

The main objective of this paper is to perform a comparison among ML, IMH, RWM and Slice sampling (SS) methods in the estimation of Weibull distribution parameters. For this, a numerical simulation study was carried out to investigate behavior of each estimation method considering different sample sizes and percentages of censuring. Based on sample root mean square errors (RMSE) and bias, we determine which method provides superior estimates of the parameters. Besides, we compare the performance of the Bayesian computational methods (IMH, RWM and SS) using the Effective sample size and the Integrated autocorrelation time. Results obtained shows that SS can be an effective alternative to standard MCMC methods to get samples from posterior distributions of parameters of the Weibull distribution. We also apply the four methods to three available real data sets.

The remainder of the paper is organized as follows. In Section 2, we introduce the Weibull model and the estimation method via Maximum Likelihood function. In Section 3, we develop the Bayesian approach and present the resampling procedures via Independent Metropolis-Hastings, Random Walk Metropolis and Slice sampling. In Section 4, a simulation study is provided. In Section 5, we apply the methods on three real datasets. Finally, in Section 6 we summarize our findings.

2. The Two-parameters Weibull Model and Observed Data

Let $T_1, \ldots, T_n$ are independent and identically distributed Weibull random variables, *i.e.*,

$$T_1, \ldots, T_n \overset{iid}{\sim} W(\alpha, \beta), \tag{1}$$

with shape parameter α and scale parameter $\beta^{-1/\alpha}$, each one having probability density function

$$f(t_i|\alpha,\beta) = \beta\alpha t_i^{\alpha-1} exp\{-\beta t_i^{\alpha}\},$$

for $t_i > 0$, $\alpha > 0$, $\beta > 0$ and $i = 1, \ldots, n$.

The survival function $S(t_i|\alpha,\beta)$ and hazard function $h(t_i|\alpha,\beta)$ are given by

$$S(t_i|\alpha,\beta) = exp\{-\beta t_i^{\alpha}\} \qquad \text{and} \qquad h(t_i|\alpha,\beta) = \beta\alpha t_i^{\alpha-1},$$

respectively, for $i = 1, \ldots, n$.

If parameter α is less than 1 the hazard function is decreasing with t. Whereas if α is greater than 1 the hazard function is increasing with t; and if $\alpha = 1$, the hazard function is constant. For this last case, the Weibull distribution becomes the Exponential distribution with parameter β.

We shall now assume that we observe possibly right censored data for n individuals. Thus, let $\mathbf{y} = (y_1, \ldots, y_n)$ be the observed dataset, where $y_i = (t_i, \delta_i)$, t_i is an observed time point and δ_i is a indicator function given by

$$\delta_i = \begin{cases} 1, \text{ if the lifetime is uncensored, } i.e., \ T_i = t_i; \\ 0, \text{ if the lifetime is censored, } i.e., \ T_i > t_i \end{cases}$$

for $i = 1, \ldots, n$.

Given $\mathbf{y}$, the likelihood function for (α, β) takes the form

$$L(\alpha, \beta|\mathbf{y}) = \prod_{i=1}^{n} \left[f(t_i|\alpha, \beta)\right]^{\delta_i} \left[S(t_i|\alpha, \beta)\right]^{1-\delta_i} = \beta^r \alpha^r exp\left\{\alpha \sum_{i=1}^{n} \delta_i log(t_i) - \beta \sum_{i=1}^{n} t_i^{\alpha}\right\} \quad (2)$$

where $r = \sum_{i=1}^{n} \delta_i$ is the amount of uncensored data. A demonstration of the equation (2) is given by Lawless (1974), p.53.

2.1 Maximum Likelihood Estimation

The method of maximum likelihood estimation is a commonly used procedure for estimating parameters. The maximum likelihood estimates (MLE) $\hat{\alpha}$ and $\hat{\beta}$ of the parameters α and β maximize function (2) or, equivalently, the logarithm likelihood function

$$l(\alpha, \beta|\mathbf{y}) = rlog(\beta) + rlog(\alpha) + \alpha \sum_{i=1}^{n} \delta_i log(t_i) - \beta \sum_{i=1}^{n} t_i^{\alpha}. \quad (3)$$

Differentiating (3) with respect to α and β, respectively, yields

$$\frac{\partial l(\alpha, \beta|\mathbf{y})}{\partial \alpha} = \frac{r}{\alpha} + \sum_{i=1}^{n} \delta_i log(t_i) - \beta \sum_{i=1}^{n} t_i^{\alpha} log(t_i), \quad (4)$$

$$\frac{\partial l(\alpha, \beta|\mathbf{y})}{\partial \beta} = \frac{r}{\beta} - \sum_{i=1}^{n} t_i^{\alpha}. \quad (5)$$

Equalling (4) and (5) to zero, the MLE $(\hat{\alpha}, \hat{\beta})$ of (α, β) are solutions of the following equations

$$\hat{\beta} = \frac{r}{\sum_{i=1}^{n} t_i^{\hat{\alpha}}} \quad \text{and} \quad \frac{r}{\hat{\alpha}} - \hat{\beta} \sum_{i=1}^{n} t_i^{\hat{\alpha}} log(t_i) = \sum_{i=1}^{n} \delta_i log(t_i). \quad (6)$$

Equations in (6) do not have explicit solutions. Therefore, we apply numerical methods to solve these equations. Iterative solutions of these equations are the MLE of the parameters α and β. We obtain the MLE using the statistical R software (R Core Team, 2017) and the *optim* command.

3. Bayesian Inference

In order to develop the Bayesian approach we need to specify the prior distributions for α and β. Firstly, we assume that priors are independent, *i.e.*, $\pi(\alpha,\beta)=\pi(\alpha)\pi(\beta)$. So, we consider the following prior distributions

$$\alpha|a_1,a_2\sim\Gamma(a_1,a_2)\qquad\text{and}\qquad\beta|b_1,b_2\sim\Gamma(b_1,b_2)$$

where $\Gamma(\cdot)$ represents the Gamma distribution and a_1, a_2, b_1 and b_2 are known hyperparameters all of them with support on $(0,+\infty)$. The choice of the hyperparameters values will generally depend upon the application at hand. At this moment, we leave them unspecified.

Using the Bayes theorem, the joint posterior distribution for (α,β) upon which inference is based, is given by

$$\pi(\alpha,\beta|\mathbf{y})\propto L(\alpha,\beta|\mathbf{y})\pi(\alpha)\pi(\beta),$$

where $L(\alpha,\beta|\mathbf{y})$ is given in (2).

The conditional posterior distributions are

$$\beta|\mathbf{y},\alpha\sim\Gamma\left(b_1+r,b_2+\sum_{i=1}^{n}t_i^{\alpha}\right)\tag{7}$$

and

$$\alpha|\mathbf{y},\beta\propto\alpha^{a_1+r-1}\exp\left\{-\alpha\left[a_2-\sum_{i=1}^{n}\delta_i log(t_i)\right]-\beta\sum_{i=1}^{n}t_i^{\alpha}\right\}.\tag{8}$$

Note from (7), that the resampling procedure for β is given in a straightforward way via Gibbs sampling algorithm.

Gibbs sampling algorithm: Let the current state of the Markov chain consist of $\left(\alpha^{(l-1)},\beta^{(l-1)}\right)$, where l is the l-th iteration of the algorithm, for $l=1,\ldots,L$. So, generate $\beta^{(l)}\sim\Gamma\left(b_1+r,b_2+\sum_{i=1}^{n}t_i^{\alpha^{(l-1)}}\right)$, where $\alpha^{(l-1)}$ is the value of α in $(l-1)$-th iteration, in which, $\alpha^{(0)}$ is the initial value of α.

However, the conditional posterior distribution for α does not follow any closed distribution and its resampling procedure is not given in a straightforward way. For this case, the usual Bayesian procedure is to use the Metropolis-Hastings (MH) algorithm.

3.1 Metropolis-Hastings algorithm

The Metropolis-Hastings algorithm together with the Gibbs sampling are the two most popular example of a Markov chain Monte Carlo (MCMC) method. This algorithm is used for sampling from some generic distribution that we do not know how to generate a random sample. Similar to acceptance-rejection sampling, the MH algorithm consider that, to each iteration of the algorithm, a candidate value can be generated from a proposal distributions. So, the candidate value is accepted according to an adequate acceptance probability. This procedure guarantees the convergence of the Markov chain for the target density. For more details on MH algorithm see Hastings (1970), Chib and Greenberg (1995), Gelman et al. (1995) and Gilks et al. (1996).

In order to update parameter α via MH algorithm, consider (α, β) be the current state of the Markov chain. Let α^* to be a candidate value generated from a candidate generating-density $q[\alpha^*|\alpha]$. So, the value α^* is accepted with probability $\Psi(\alpha^*|\alpha) = min(1, A_\alpha)$, where

$$A_\alpha = \frac{L(\alpha^*, \beta|\mathbf{y})\pi(\alpha^*)}{L(\alpha, \beta|\mathbf{y})\pi(\alpha)} \frac{q[\alpha|\alpha^*]}{q[\alpha^*|\alpha]} \tag{9}$$

and $L(\cdot|\mathbf{y})$ is the likelihood function, given in equation (2).

In practical terms, the MH algorithm is implemented as follows.

Metropolis-Hastings algorithm: Let the current state of the Markov chain consist of $\left(\alpha^{(l-1)}, \beta^{(l)}\right)$, where l is the l-th iteration of the algorithm, for $l = 1, \ldots, L$. So, update α as follows:

(1) Generate $\alpha^* \sim q[\alpha^*|\alpha]$;
(2) Calculate $\Psi(\alpha^*|\alpha) = min(1, A_\alpha)$, where A_α is given in (9);
(3) Generate $u \sim U(0, 1)$. If $u \leq \Psi(\alpha^*|\alpha)$ accept α^* and do $\alpha^{(l)} = \alpha^*$. Otherwise, reject α^* and do $\alpha^{(l)} = \alpha^{(l-1)}$.

3.1.1 Two common choices for $q[\cdot]$

To implement the MH algorithm, it is necessary specify the candidate-generating density $q[\alpha^*|\alpha]$. Usually $q[\cdot]$ is chosen such that it is easy to sample from it. In this section, we describe the two common choice of $q[\cdot]$.

- *Independent Metropolis-Hastings (IMH).* If $q[\alpha^*|\alpha] = q[\alpha^*]$, *i.e.*, the candidate generating-density does not depend on the current α value, then we get a special case of the original algorithm, in which, A_α simplifies to

$$A_\alpha = \frac{L(\alpha^*, \beta|\mathbf{y})\pi(\alpha^*)}{L(\alpha, \beta|\mathbf{y})\pi(\alpha)} \frac{q[\alpha]}{q[\alpha^*]}.$$

In order to implement this case, one may set $q[\alpha^*]$ as the prior distribution, *i.e.*, $q[\alpha^*] = \pi(\alpha^*)$. So, A_α is given by the likelihood ratio

$$A_\alpha = \frac{L(\alpha^*, \beta|\mathbf{y})}{L(\alpha, \beta|\mathbf{y})}. \tag{10}$$

This algorithm is implemented as follows.

Independent Metropolis-Hastings algorithm: Let the current state of the Markov chain consist of $\left(\alpha^{(l-1)}, \beta^{(l)}\right)$. For l-th iteration of the algorithm, $l = 2, \ldots, L$:

(1) Generate $\alpha^* \sim \Gamma(a_1, a_2)$;
(2) Calculate $\Psi(\alpha^*|\alpha) = min(1, A_\alpha)$, where A_α is given in (10);
(3) Generate $u \sim U(0, 1)$. If $u \leq \Psi(\alpha^*|\alpha)$ accept α^* and do $\alpha^{(l)} = \alpha^*$. Otherwise, reject α^* and do $\alpha^{(l)} = \alpha^{(l-1)}$.

Although the choice of the prior distributions as the candidate generating-density is mathematically attractive, this may lead to many rejections of the proposed moves and a slow convergence of the algorithm. This happen, specially, for cases in which no prior information is available and prior distribution has large variance. An alternative is to explore the neighbourhood of the current value of the chain in order to propose a new value. This method is called by random walk Metropolis.

- *Random walk Metropolis (RWM).* If the candidate generating-density $q[\alpha^*|\alpha]$ is symmetric and the probability of generating a move from α to α^* depends only on the distance between them, *i.e.*, $q[\alpha^*|\alpha] = g(|\alpha - \alpha^*|)$, where g is a symmetric density, then A_α simplifies to

$$A_\alpha = \frac{L(\alpha^*, \beta|\mathbf{y})\pi(\alpha^*)}{L(\alpha, \beta|\mathbf{y})\pi(\alpha)} \tag{11}$$

since the proposal kernels from numerator and denominator cancel.

In order to implement this case, we simulate α^* setting $\alpha^* = \alpha + \varepsilon$, where ε is a random perturbation generated from a Normal distribution with mean equals to 0 and variance σ^2_α, $\varepsilon \sim \mathcal{N}(0, \sigma^2_\alpha)$, meaning that $\alpha^* \sim \mathcal{N}(\alpha, \sigma^2_\alpha)$. This algorithm is implemented as follows.

Random walk Metropolis algorithm: Let the current state of the Markov chain consist of $\left(\alpha^{(l-1)}, \beta^{(l)}\right)$. For l-th iteration of the algorithm, $l = 1, \ldots, L$:

(1) Generate $\varepsilon \sim \mathcal{N}(0, \sigma^2_\alpha)$ and set $\alpha^* = \alpha^{(l-1)} + \varepsilon$;
(2) Calculate $\Psi(\alpha^*|\alpha) = min(1, A_\alpha)$, where A_α is given in (11);
(3) Generate $u \sim U(0, 1)$. If $u \leq \Psi(\alpha^*|\alpha)$ accept α^* and do $\alpha^{(l)} = \alpha^*$. Otherwise, reject α^* and do $\alpha^{(l)} = \alpha^{(l-1)}$.

As discussed by Chib and Greenberg (1995) an important issue on RWM is how to fix the value of σ^2_α. This value has great influence on efficiency of the algorithm. If σ^2_α is small, then random perturbations will be small in magnitude and almost all will be accepted. As a consequence, it will require a large number of iterations to explore the whole space-state. On the other hand, if σ^2_α is too large, then it will causes too many rejections of the proposed moves and a considerably slowing down convergence. More details on these facts can be found in Roberts et al. (1997); Bedard (2007); Mattingly et al. (2011).

Typically, one may fix the value of σ^2_α testing some values in a few pilot runs and then choosing a value in which acceptance ratio lies in between 20% and 30% (see, for example, Gilks et al. (1996) and Gelman et al. (1995)). But, in the most of cases this makes the method very tedious.

3.2 Slice sampling algorithm

An alternative to the MH for sampling from some generic distribution is the Slice sampling algorithm, proposed by Neal (2003). This algorithm is a kind of Gibbs sampling based on the simulation of specific Uniform random variables.

For this, an auxiliary variable U is introduced and the jointly distribution $\pi(\alpha, U|\mathbf{y}, \beta)$ is given by a Uniform distribution over region $\mathrm{U} = \{(\alpha, u) : 0 < u < h(\alpha)\}$ below the curve on surface defined by $h(\alpha)$, where from (8)

$$h(\alpha) = \alpha^{a_1+r-1} \exp\left\{-\alpha\left[a_2 - \sum_{i=1}^{n} \delta_i log(t_i)\right] - \beta \sum_{i=1}^{n} t_i^\alpha\right\}.$$

Since, marginalizing $\pi(\alpha, u|\mathbf{y}, \beta)$ over u yields $\pi(\alpha|\mathbf{y}, \beta)$, sampling from $\pi(\alpha, u|\mathbf{y}, \beta)$ and discarding u is equivalent to sampling from $\pi(\alpha|\mathbf{y}, \beta)$.

As sampling from $\pi(\alpha, U|\mathbf{y}, \beta)$ is not straightforward, we implement a Gibbs sampling algorithm where in every iteration l, we first sample $U^{(l)} \sim \mathcal{U}\left(0, h(\alpha^{(l-1)})\right)$ and then sample $\alpha^{(l)} \sim \mathcal{U}(A)$, where $A = \{\alpha : u^{(l)} < h(\alpha)\}$. However, as inverse of $h(\alpha)$ can not be obtained analytically, we adapt the following procedure to update α:

(i) let $\lambda = 0.01$ and $\tilde{A}$ be an empty set. For $m = 1, 2, \ldots$ do the following:
  (a) set $\alpha^{-(m)} = \alpha^{(l-1)} - m\lambda$. If $u^{(l)} < h\left(\alpha^{-(m)}\right)$ do $\tilde{A} = \tilde{A} \cup \left\{\alpha^{-(m)}\right\}$. If $u^{(l)} \geq h\left(\alpha^{-(m)}\right)$, stop the procedure. Go to item (b);
  (b) set $\alpha^{+(m)} = \alpha^{(l-1)} + m\lambda$. If $u^{(l)} < h\left(\alpha^{+(m)}\right)$ do $\tilde{A} = \tilde{A} \cup \left\{\alpha^{+(m)}\right\}$. If $u^{(l)} \geq h\left(\alpha^{+(m)}\right)$, stop the procedure. Go to item (ii);

(ii) Generate $\alpha^{(l)} \sim \mathcal{U}(min(\widetilde{A}), max(\widetilde{A}))$.

This algorithm is implemented as follows.

Slice sampling algorithm: Let the current state of the Markov chain consist of $\left(\alpha^{(l-1)}, \beta^{(l)}\right)$ and $u^{(l-1)}$. For l-th iteration of the algorithm, $l = 1, \ldots, L$:

(1) Generate $U^{(l)} \sim \mathcal{U}\left(0, h(\alpha^{(l-1)})\right)$;
(2) Conditional on $u^{(l)}$ get $\widetilde{A}$ as described in step (i) above;
(3) Generate $\alpha^{(l)} \sim \mathcal{U}(min(\widetilde{A}), max(\widetilde{A}))$;

4. Simulation study

In order to illustrate and compare performance of methods described above, a random sample of size $n = 25, 50, 100$ and 200 with 0%, 10%, 20% and 30% of times right-censored were generated from the Weibull distribution to take care of small, medium and large data sets.

The shape parameter α was chosen to be 0.5, 1 and 2 in order to get a decreasing, constant and increasing hazard function, respectively. The scale parameter β was fixed equals to 1, $\beta = 1$. Appendix A shows results for $\alpha = \beta = 2$. For each value of α and β we generate $M = 1,000$ different artificial data sets and the parameters were estimated using MLE, IMH, RWM and SS. We fix hyperparameters values as $a_1 = a_2 = b_1 = b_2 = 0.1$ in order to get a prior distribution with large variance.

For m-th generated artificial data set, we apply IMH, RWM and SS fixing $L = 55,000$ iterations and the *burn in* $B = 5,000$. We also consider jumps of size 10, *i.e.*, only 1 draw from every 10 was kept, in order to construct a sample of size $5,000$ to make inferences. According with these values the estimates $\left(\tilde{\alpha}^{(m)}, \tilde{\beta}^{(m)}\right)$ for (α, β) are given by the average of the values generated, *i.e.*,

$$\tilde{\alpha}^{(m)} = \frac{1}{L}\sum_{l=1}^{L} \alpha^{(l)} \qquad \text{and} \qquad \tilde{\beta}^{(m)} = \frac{1}{L}\sum_{l=1}^{L} \beta^{(l)},$$

for $m = 1, \ldots, M$. We present results using the average of the M parameters estimates, denoted by $\overline{\alpha}$ and $\overline{\beta}$. The comparisons among methods is done in terms of the sample Root Mean Square Error (RMSE), given by

$$RMSE = \sqrt{\frac{1}{M}\sum_{m=1}^{M}\left[(\hat{\alpha}^{(m)} - \alpha)^2 + (\hat{\beta}^{(m)} - \beta)^2\right]}$$

and by mean of the sum of the modulus of the bias, given by

$$MBias = \frac{1}{M}\sum_{m=1}^{M}\left[|\hat{\alpha}^{(m)} - \alpha| + |\hat{\beta}^{(m)} - \beta|\right].$$

A smaller $RMSE$ and $MBias$ indicates a better overall quality of the estimates.

Tables 1, 2 and 3 present the average of estimates (Est.) and the $RMSE$ and $MBias$ values by method, for $\alpha = 0.5$, 1 and 2, respectively. In these Tables we also present the percentage of values accepted (% acc.) for IMH and RWM. The smaller $RMSE$ and $MBias$ for each sample size and percentage of censuring are highlighted in bold.

Table 1. Average of estimates, $RMSE$ and $MBias$ by method. True parameters are $\alpha = 0.5$ and $\beta = 1$.

Method	Est.	Sample Size															
		25				50				100				200			
		0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%
MLE	$\overline{\alpha}$	0.530	0.529	0.529	0.532	0.516	0.514	0.517	0.518	0.506	0.504	0.505	0.505	0.505	0.505	0.595	0.505
	$\overline{\beta}$	1.025	1.026	1.029	1.037	1.002	1.004	1.009	1.014	1.003	1.001	1.003	1.003	1.002	1.004	1.003	1.004
	$RMSE$	0.250	0.265	0.284	0.304	0.168	0.181	0.192	0.203	0.121	0.129	0.138	0.143	0.080	0.093	0.092	0.097
	$MBias$	0.247	0.266	0.282	0.305	0.173	0.187	0.202	0.213	0.119	0.130	0.140	0.149	0.082	0.088	0.0096	0.102
IMH	$\overline{\alpha}$	0.525	0.525	0.534	0.528	0.510	0.510	0.508	0.512	0.505	0.504	0.506	0.508	0.505	0.502	0.502	0.504
	$\overline{\beta}$	1.029	1.026	1.011	1.046	1.014	1.001	1.010	1.012	1.007	1.008	1.002	1.004	0.999	0.994	0.999	1.006
	$RMSE$	0.251	0.258	0.284	0.309	0.172	0.176	0.187	0.204	0.118	0.131	0.134	0.140	0.084	0.086	0.095	0.101
	$MBias$	0.246	0.269	0.293	0.308	0.175	0.183	0.193	0.210	0.123	0.136	0.141	0.147	0.089	0.092	0.096	0.106
	$\%acc$	0.521	0.623	0.680	0.726	0.413	0.474	0.511	0.538	0.323	0.376	0.401	0.422	0.265	0.302	0.330	0.339
RWM	$\overline{\alpha}$	0.535	0.526	0.524	0.528	0.512	0.513	0.513	0.513	0.512	0.505	0.508	0.505	0.503	0.502	0.503	0.503
	$\overline{\beta}$	1.020	1.027	1.020	1.034	1.013	1.007	1.021	1.010	1.004	1.009	1.005	1.017	1.003	1.003	1.008	1.003
	$RMSE$	0.250	0.260	**0.266**	**0.303**	0.171	0.178	**0.186**	**0.195**	0.114	0.125	**0.131**	**0.138**	**0.079**	**0.085**	**0.089**	**0.095**
	$MBias$	0.251	0.261	**0.271**	**0.301**	0.172	0.181	**0.190**	**0.199**	0.117	0.129	**0.134**	**0.143**	**0.080**	**0.086**	**0.094**	**0.099**
	$\%acc$	9.967	11.791	12.904	13.675	6.794	8.253	9.052	9.506	4.829	5.791	6.417	6.694	3.364	4.103	4.508	4.756
SS	$\overline{\alpha}$	0.525	0.525	0.526	0.531	0.514	0.511	0.508	0.508	0.505	0.506	0.507	0.506	0.505	0.505	0.056	0.503
	$\overline{\beta}$	1.021	1.030	1.020	1.057	1.011	1.016	1.015	1.010	1.008	1.001	1.009	1.015	1.006	1.004	1.005	1.001
	$RMSE$	**0.242**	**0.256**	0.282	0.318	**0.166**	**0.175**	0.191	0.196	**0.113**	**0.124**	0.132	0.146	0.081	0.086	0.092	0.096
	$MBias$	**0.238**	**0.258**	0.280	0.316	**0.168**	**0.178**	0.194	0.201	**0.116**	**0.128**	0.136	0.150	0.082	0.087	0.096	0.101

Table 2. Average of estimates, $RMSE$ and $MBias$ by method. True parameters are $\alpha = 1$ and $\beta = 1$.

Method	Est.	Sample Size															
		25				50				100				200			
		0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%
MLE	$\overline{\alpha}$	1.059	1.063	1.064	1.065	1.031	1.032	1.032	1.036	1.012	1.010	1.010	1.009	1.009	1.009	1.009	1.009
	$\overline{\beta}$	1.026	1.029	1.032	1.035	1.003	1.006	1.009	1.013	1.002	1.002	1.003	1.002	1.003	1.003	1.002	1.003
	$RMSE$	0.302	0.322	0.343	0.370	0.199	0.209	0.223	0.243	0.137	0.143	0.153	0.164	0.095	0.099	0.105	0.116
	$MBias$	0.319	0.338	0.358	0.380	0.220	0.232	0.247	0.268	0.152	0.159	0.170	0.183	0.104	0.110	0.115	0.124
IMH	$\overline{\alpha}$	1.050	1.051	1.051	1.050	1.020	1.025	1.017	1.021	1.013	1.013	1.019	1.008	1.003	1.006	1.002	1.008
	$\overline{\beta}$	1.028	1.031	1.024	1.033	1.014	1.012	1.009	1.012	1.006	1.009	0.995	1.008	0.999	0.9988	0.999	1.004
	$RMSE$	0.297	0.321	0.336	0.356	0.205	0.212	0.221	0.241	0.146	0.147	0.159	0.174	0.104	0.112	0.111	0.121
	$MBias$	0.319	0.343	0.367	0.377	0.225	0.231	0.244	0.262	0.164	0.169	0.176	0.193	0.117	0.125	0.128	0.135
	$\%acc$	0.542	0.589	0.638	0.691	0.425	0.443	0.482	0.517	0.331	0.356	0.384	0.413	0.278	0.284	0.315	0.332
RWM	$\overline{\alpha}$	1.064	1.060	1.054	1.046	1.022	1.020	1.019	1.027	1.020	1.013	1.010	1.013	1.005	1.004	1.011	1.003
	$\overline{\beta}$	1.020	1.040	1.041	1.030	1.013	1.017	1.012	1.019	1.004	1.004	1.005	1.009	1.003	1.003	1.003	1.004
	$RMSE$	0.296	0.313	0.338	0.355	0.199	0.211	0.218	0.236	0.135	0.144	0.151	0.162	0.093	0.099	0.106	0.116
	$MBias$	0.320	0.336	0.359	0.376	0.217	0.230	0.239	0.258	0.150	0.160	0.166	0.179	0.105	0.112	0.116	0.128
	$\%acc$	19.391	20.963	22.665	24.398	13.428	14.617	15.993	15.554	9.557	10.360	11.392	12.438	6.697	7.294	8.101	8.803
SS	$\overline{\alpha}$	1.059	1.060	1.040	1.063	1.027	1.022	1.019	1.021	1.016	1.001	1.013	1.010	1.007	1.007	1.005	1.005
	$\overline{\beta}$	1.023	1.018	1.028	1.035	1.008	1.012	1.021	1.014	1.001	1.011	0.997	1.010	1.003	1.008	1.006	1.003
	$RMSE$	**0.295**	**0.308**	**0.335**	**0.354**	**0.198**	**0.207**	**0.210**	**0.234**	**0.134**	**0.134**	**0.147**	**0.158**	**0.092**	**0.097**	**0.103**	**0.115**
	$MBias$	**0.317**	**0.331**	**0.357**	**0.368**	**0.215**	**0.225**	**0.236**	**0.255**	**0.148**	**0.158**	**0.165**	**0.177**	**0.101**	**0.109**	**0.112**	**0.121**

Table 3. Average of estimates, $RMSE$ and $MBias$ by method. True parameters are $\alpha = 2$ and $\beta = 1$.

Method	Est.	Sample Size															
		25				50				100				200			
		0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%
MLE	$\overline{\alpha}$	2.119	2.132	2.143	2.152	2.062	2.066	2.073	2.077	2.024	2.023	2.024	2.014	2.018	2.018	2.018	2.019
	$\overline{\beta}$	1.026	1.029	1.036	1.041	1.003	1.006	1.011	1.013	1.002	1.003	1.004	1.005	1.003	1.004	1.002	1.003
	$RMSE$	0.456	0.487	0.521	0.559	0.291	0.306	0.325	0.346	0.199	0.208	0.221	0.237	0.138	0.144	0.151	0.159
	$MBias$	0.463	0.485	0.515	0.550	0.315	0.328	0.347	0.373	0.217	0.226	0.239	0.256	0.150	0.158	0.164	0.171
IMH	$\overline{\alpha}$	2.102	2.087	2.130	2.123	2.045	2.055	2.055	2.054	2.023	2.006	2.021	2.026	2.004	2.002	2.011	2.002
	$\overline{\beta}$	1.028	1.033	1.034	1.025	1.013	1.004	1.004	1.008	1.006	1.003	0.999	1.010	0.999	1.004	0.999	1.006
	$RMSE$	0.439	0.472	0.534	0.541	0.310	0.326	0.332	0.366	0.281	0.222	0.234	0.247	0.160	0.166	0.173	0.180
	$MBias$	0.461	0.491	0.550	0.555	0.332	0.342	0.354	0.390	0.237	0.241	0.250	0.260	0.174	0.181	0.186	0.192
	$\%acc$	0.543	0.541	0.594	0.658	0.426	0.446	0.465	0.491	0.342	0.354	0.368	0.388	0.289	0.297	0.308	0.315
RWM	$\overline{\alpha}$	2.123	2.111	2.119	2.116	2.043	2.044	2.048	2.053	2.039	2.019	2.017	2.027	2.010	2.014	2.012	2.011
	$\overline{\beta}$	1.021	1.026	1.060	1.048	1.014	1.009	1.015	1.026	1.004	1.008	1.014	1.004	1.003	1.003	1.002	1.002
	$RMSE$	**0.427**	**0.469**	**0.484**	**0.512**	**0.284**	**0.299**	**0.314**	**0.336**	**0.198**	**0.199**	**0.209**	**0.229**	**0.133**	**0.141**	**0.149**	**0.156**
	$MBias$	**0.457**	**0.478**	**0.500**	**0.535**	**0.305**	**0.320**	**0.337**	**0.362**	**0.215**	**0.218**	**0.227**	**0.246**	**0.145**	**0.156**	**0.163**	**0.168**
	$\%acc$	35.543	37.143	39.091	41.068	25.682	27.075	28.809	30.682	18.681	19.865	20.809	22.346	13.237	14.013	14.918	16.021
SS	$\overline{\alpha}$	2.133	2.104	2.120	2.150	2.049	2.072	2.067	2.076	2.034	2.025	2.031	2.026	2.016	2.016	2.011	2.018
	$\overline{\beta}$	1.018	1.039	1.037	1.042	1.007	1.002	1.014	1.016	0.997	1.004	1.001	1.005	1.004	1.003	1.002	1.002
	$RMSE$	0.458	0.470	0.503	0.557	0.287	0.300	0.321	0.338	0.202	0.206	0.215	0.231	0.139	0.145	0.150	0.157
	$MBias$	0.471	0.484	0.511	0.566	0.307	0.321	0.346	0.365	0.219	0.223	0.234	0.252	0.152	0.160	0.164	0.173

Fixing the sample size n and increasing the censuring percentage (% cens.), the values of $RMSE$ and $MBias$ increases for all methods. In the other hand, when the number of sample size increases, for censuring percentage fixed, the $RMSE$ and $MBias$ values decreases in all cases and hence the estimation precision of the parameters increases.

Results shows that in general the $RMSE$ and $MBias$ values of the RWM or SS outperforms MLE and IMH. Besides, as we can note RWM and SS present a complementarity. For $\alpha = 0.5$, the SS presents smaller $RMSE$ and $MBias$ for 0% and 10% of censuring and $n = \{25, 50, 100\}$; while RWM presents smaller $RMSE$ and $MBias$ for 20% and 30% and all sample size considered. For $\alpha = 1$, the SS presents smaller $RMSE$ and $MBias$; while for $\alpha = 2$, the RWM presents smaller $RMSE$ and $MBIas$.

Besides, note that the percentage of acceptance from RWM is satisfactorily; while the percentage of the IMH is very low indicating a slow convergence of the method. We verify the convergence of IMH, RWM and SS using the effective sample size (Kass et al., 1998) and integrated autocorrelation time (IAT). The effective sample size (ESS) is the number of effectively independent draws from posterior distribution. The method with larger ESS is the most efficient. IAT is a MCMC diagnostic which estimates the number of autocorrelated samples, on average, required to produce one independent draws sample. The method with the lowest IAT is the most efficient. The EES and IAT values were obtained using the coda and LaplacesDemon packages in the R software.

Tables 11, 12 and 13 in Appendix B show the average of ESS and IAT by method for $\alpha = 0.5, 1, 2$, respectively. For all simulated cases SS present better performance than IMH and RWM, *i.e.*, higher ESS and smaller IAT average values than IMH and RWM. These results show us that SS can be an effective alternative in relation to IMH and RWM to get MCMC samples from posterior distributions of Weibull distribution parameters when Gamma prior distributions are assumed. In appendix C we present an empirical illustration of the convergence of the IMH, RWM and SS.

5. Application

In this Section, we examine the performance of MLE, IMH, RWM and SS on three publicly available data sets. The first one is a dataset on the times of failure described in Whitmore (1983). The second one is a Leukemia data set downloaded from website https://docs.ufpr.br/~giolo/Livro/ApendiceA/leucemia.txt. The third one dataset refers to a clinical study carried out with 28 patients with cancer of the head and neck that did not respond to chemotherapy. This dataset is described by Lee and Wang (2003). In order to compare performance of methods we consider the RMSE in relation to empirical distribution function, given by

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[\hat{F}(x_i) - F(x_i)\right]^2},$$

where $\hat{F}(x)$ is obtained by substituting the estimates of α and β (obtained by each method) while $F(x_i)$ is the empirical distribution function obtained from Kaplan-Meier estimates, for $i = 1, \ldots, n$. The method with the minimum RMSE becomes the best method for the estimation of Weibull parameters among the candidate methods.

5.1 Aluminium reduction cells dataset

Whitmore (1983) describe a data set on the times of failure of 20 aluminium reduction cells. This data set is also presented by Lawless (1974). The dataset has $n = 20$ times and 3 (15%) censured times. We consider that these failure times follows a Weibull distribution with parameters α and β as in model (1).

Table 4 shows the parameters estimates and RMSE values by method. For this data set, the SS method presented smaller RMSE values than other methods.

Table 4. Parameters estimates and RMSE by method.

Parameter	Method			
	MLE	IMH	RWM	SS
α	2.57623	2.56527	2.54808	2.52090
β	0.23570	0.24831	0.25082	0.25404
$RMSE$	0.00133	0.00082	0.00076	0.00071

Figure 1 shows the estimated survival function by MLE and SS. The graphics of estimated survival by IMH and RWM are very close to SS. Due this, to maintain a good visualization we display only the graphic of the estimated survival by MLE and SS. In this Figure the step function is the Kaplan-Meyer estimates. The Kaplan-Meyer estimates were obtained using the *survival* package and the *survfit* command of the *R* software.

Table 5 shows the ESS and IAT values for IMH, RWM and SS. For this dataset, SS present better performance than IMH and RWM, *i.e.*, highest ESS value and smallest IAT value by parameter.

Table 5. ESS and IAT values from IMH, RWM and SS.

Parameter	ESS			IAT		
	I MH	RWM	SS	IMH	RWM	SS
α	185.2412	1878.3580	3001.0000	149.9382	12.7126	4.5945
β	302.7919	2164.6850	3001.0000	75.2247	8.6047	4.1007

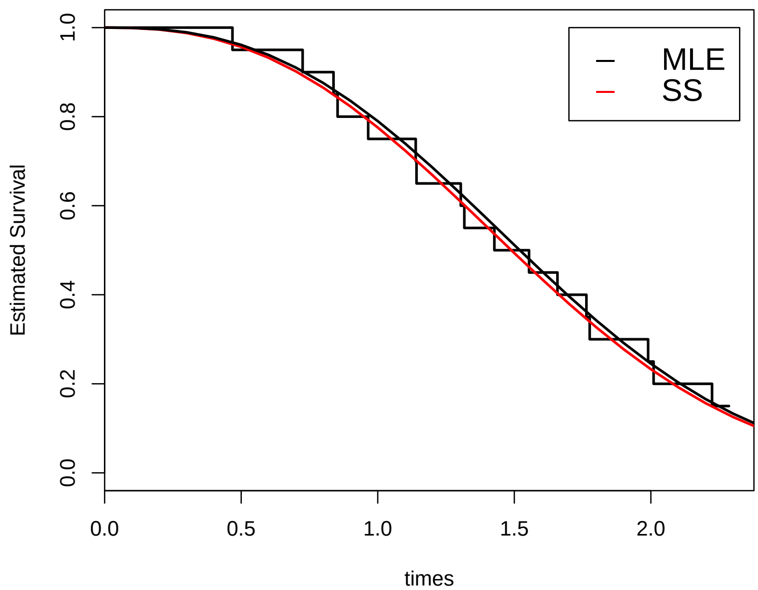


Figure 1. Estimated Survival by MLE and SS.

Figure 2 shows the traceplot and the ergodic mean for generated α values by method. Figure 3 shows the autocorrelation time. As one can note in these Figures, the α values generated by RWM and SS are well mixed and present satisfactory stability for ergodic mean and satisfactory autocorrelation. In the other hand, IMH present a poor mixing and not satisfactory autocorrelation. These results together with RMSE value show us that for this dataset SS present better performance than MLE, IMH and RWM.

5.2 Leukemia dataset

The second real data set considered is the Leukemia dataset available on website https://docs.ufpr.br/ giolo/Livro/ApendiceA/. This dataset has $n = 103$ patients with 39 (37.86%) of censured times. For more details see Colosimo and Giolo (2006).

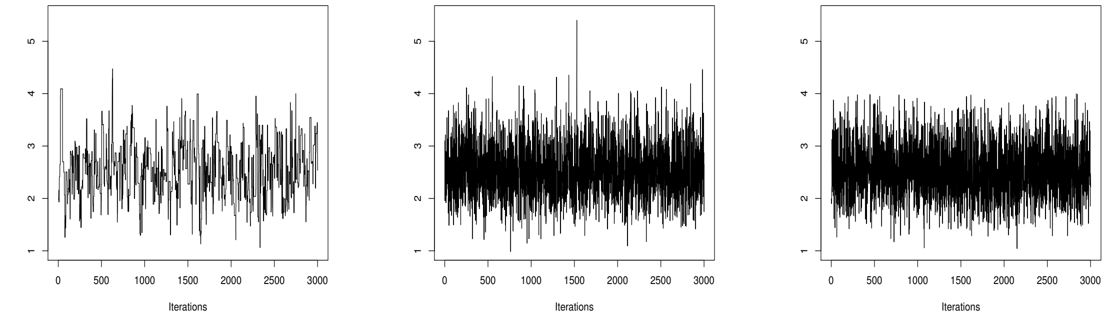
Table 6 shows the parameters estimates and RMSE values by method. For this dataset the four methods presented similar results; with a slight advantage for Bayesian methods.

Table 6. Parameters estimates and RMSE by method.

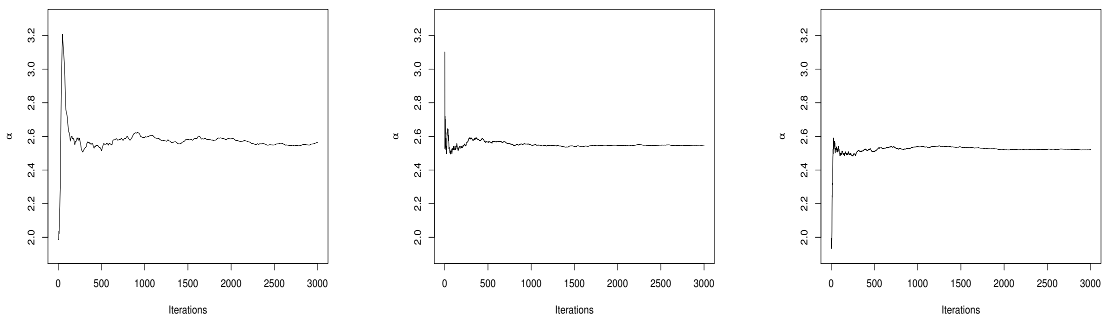
Parameter	Method			
	MLE	IMH	RWM	SS
α	1.05635	1.05038	1.05232	1.05337
β	0.20683	0.20793	0.20855	0.20840
$RMSE$	0.00042	0.00041	0.00041	0.00041

Figure 4 shows the estimated survival function by SS. As graphics of the four methods are very close between themselves, so we display only the graphic of the estimated survival by SS in order to maintain a good visualization.

Table 7 shows the ESS and IAT values for IMH, RWM and SS. Figures 5 and 6 show the traceplot, the ergodic mean and autocorrelation time for generated α values by method. As we can note, in opposite to RWM and SS, the sample generated by IMH does not mix well, the ergodic mean does not present satisfactory stability as well as autocorrelation. Thus, as example 5.1 described earlier, we have that sampled values by SS present better

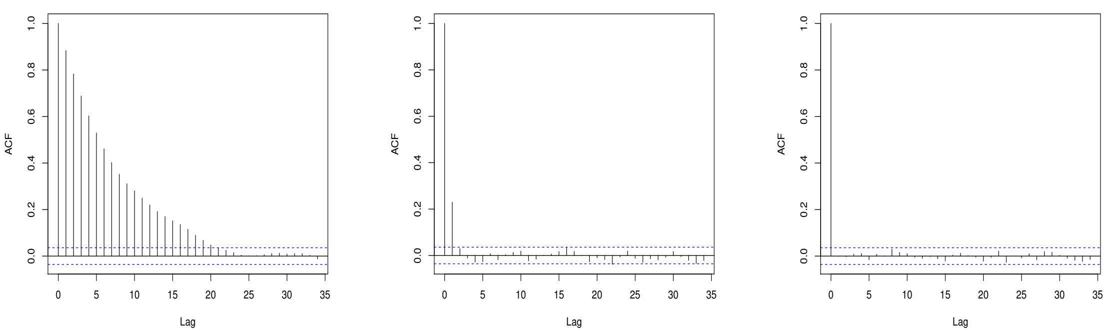


(a) IMH. (b) RWM. (c) SS.



(d) IMH. (e) RWM. (f) SS.

Figure 2. Traceplots and ergodic mean by method.



(a) IMH. (b) RWM. (c) SS.

Figure 3. Autocorrelation by method.

performance than IMH and RWM, *i.e.*, satisfactorily stability and autocorrelation with higher ESS and smaller IAT values.

Table 7. ESS and IAT values from IMH, RWM and SS.

Parameter	ESS			IAT		
	IMH	RWM	SS	IMH	RWM	SS
α	327.2075	1801.3870	3001.0000	87.2039	15.6022	2.0785
β	861.9586	2281.6410	3131.1140	21.1406	6.6192	2.0715

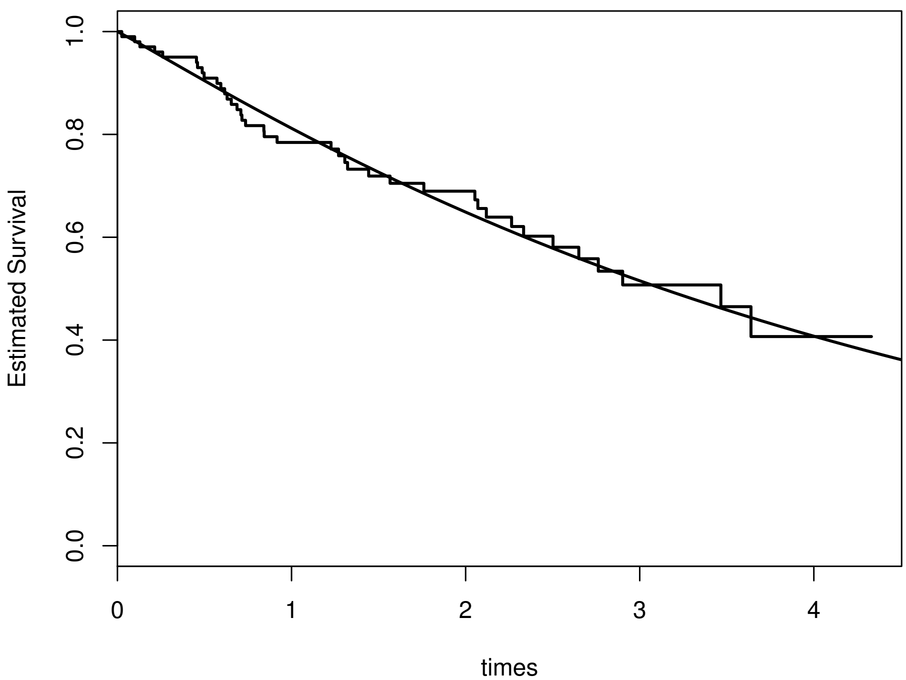
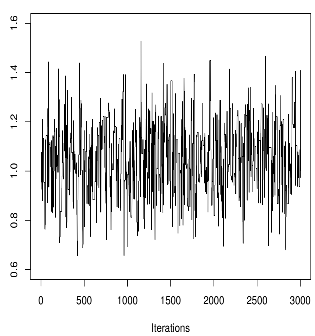
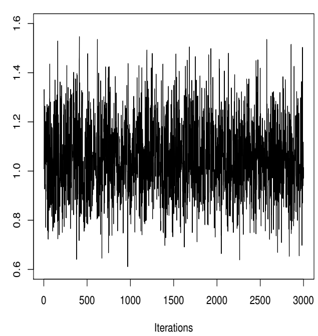


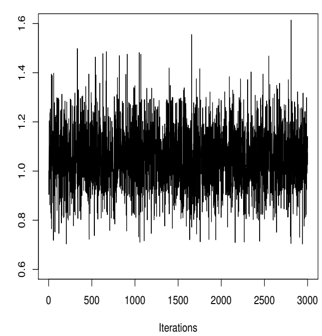
Figure 4. Estimated Survival by SS.



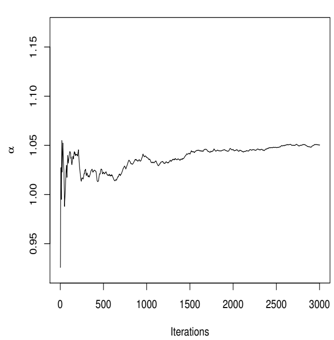
(a) IMH.



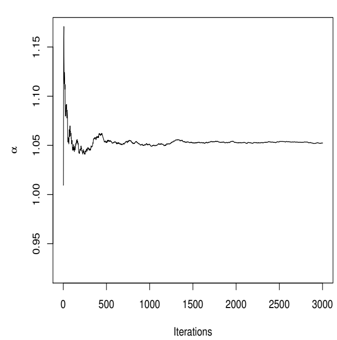
(b) RWM.



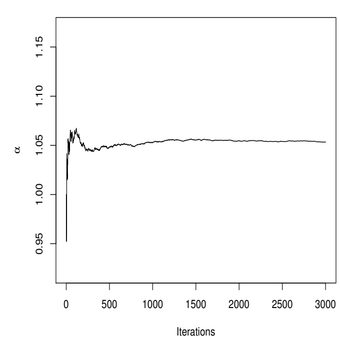
(c) SS.



(d) IMH.

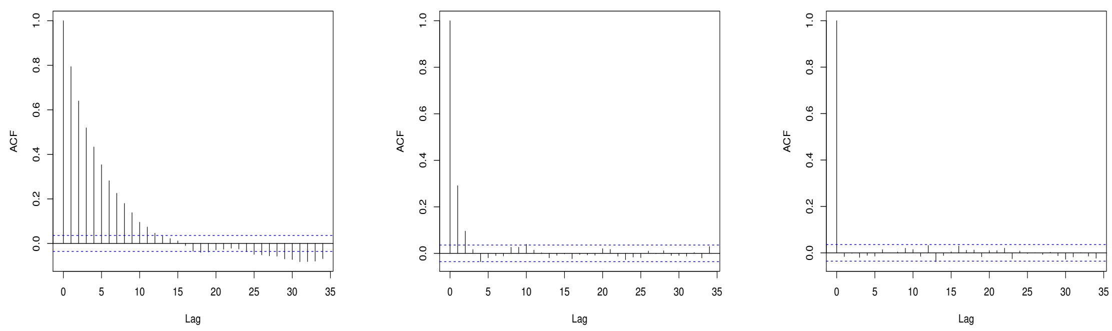


(e) RWM.



(f) SS.

Figure 5. Traceplots and Ergodic mean by method.



(a) IMH. (b) RWM. (c) SS.

Figure 6. Autocorrelation by method.

5.3 Head and Neck dataset

The Head and Neck dataset described in Lee and Wang (2003) refers to survival times, in weeks, of $n = 28$ patients with cancer of the head and neck that did not respond to chemotherapy. The dataset has 11 (39.29%) censured times.

Table 8 shows the parameters estimates and RMSE values by method. For this data set, the MLE presented smallest RMSE value. But, SS again present smaller RMSE value than IMH and RWM.

Table 8. Parameters estimates and RMSE by method.

Parameter	Method			
	MLE	IMH	RWM	SS
α	2.04158	2.02477	1.98320	1.97094
β	0.00315	0.00460	0.00545	0.00535
$RMSE$	0.00149	0.00506	0.00760	0.00470

Figure 7 shows the estimated survival function by MLE and SS. The graphics of estimated survival by IMH and RWM are very close to SS, so due this, to maintain a good visualization we display only the graphic of the estimated survival by MLE and SS.

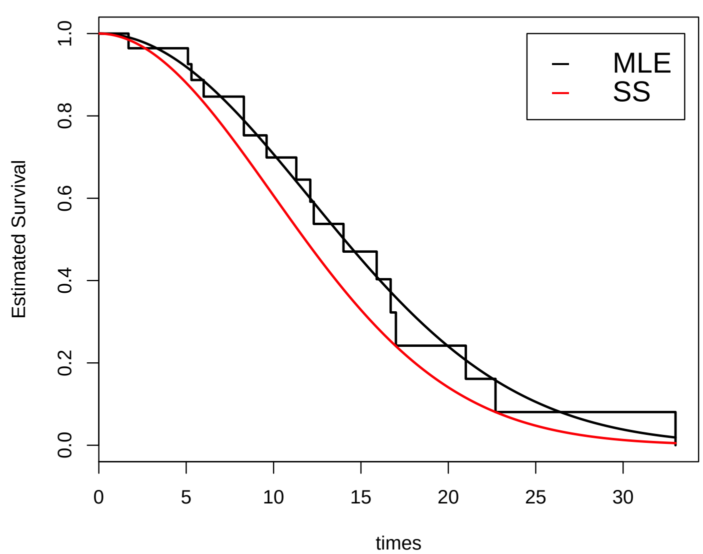


Figure 7. Estimated Survival by MLE and SS.

Table 9 shows the ESS and IAT values for IMH, RWM and SS. For this data set, SS also present better performance than IMH and RWM. The traceplot, ergodic mean and the autocorrelation times are similar to graphics presented in earlier examples.

Table 9. ESS and IAT values from IMH, RWM and SS.

Parameter	ESS			IAT		
	IMH	RWM	SS	IMH	RWM	SS
α	13.8061	125.3216	926.3438	2609.3300	231.7020	30.5874
β	19.8969	192.9900	1194.3350	97.6168	88.4905	25.8853

6. Final Remarks

In this paper we described four methods for estimating the Weibull distribution parameters in presence of right censored times. The first one is the standard Maximum likelihood estimation and the others three are Bayesian computational methods namely independent Metropolis-Hastings, random walk metropolis and Slice sampling. The performance of these methods were compared using the Monte Carlo simulation based on the RMSE criterion and bias. The RMSE and bias were calculated for different sample size and percentages of censures. We illustrate the application of the methods using three real datasets available on the literature.

Based on RMSE criterion and bias, results from artificial show a complementarity between RWM and SS. Besides, results obtained report evidence that MCMC samples got with SS has better properties (e.g. higher ESS and smaller IAT values) than IMH and RWM, which are standard methods to get samples from posterior distribution.

These results show that SS can be an effective alternative to standard method of estimation of Weibull distribution parameters. Moreover, two advantages of the SS is that method it is kind of Gibbs sampling and therefore easy to be implemented and do not need to specify a candidate-generating density. A disadvantage, is the time simulation that is greater than IMH and RWM. It is due the way that we implement SS, due the function $h(\alpha)$ do not have explicit inverse. The source code used in the simulation study and applications was implemented in R software and can be obtained by emailing the authors.

Acknowledgements

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7. Appendix A: Simulation Results for $\alpha = \beta = 2$

In this Appendix we present the average of estimates and the $RMSE$ and $MBias$ values by method, for simulated datasets with $\alpha = \beta = 2$. Estimates by method are in Table 10. This Table also present the percentage of values accepted (%acc.) for IMH and RWM. The smaller $RMSE$ and $MBias$ for each sample size and percentage of censuring are highlighted in bold. Similar to Table 3 ($\alpha = 2$ and $\beta = 1$), for this case, RWM present smaller RMSE values than others three methods considered.

Table 10. Average of estimates, $RMSE$ and $MBias$ by method. True parameters are $\alpha = 2$ and $\beta = 2$.

Method	Est.	Sample Size															
		25				50				100				200			
		0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%	0%	10%	20%	30%
MLE	$\overline{\alpha}$	2.119	2.143	2.157	2.193	2.062	2.073	2.086	2.014	2.024	2.024	2.027	2.033	2.018	2.018	2.023	2.027
	$\overline{\beta}$	2.001	2.002	2.019	2.057	2.010	2.007	2.007	2.020	2.004	2.005	2.007	2.012	2.002	2.002	2.003	2.003
	$RMSE$	0.445	0.503	0.596	0.763	0.298	0.321	0.371	0.467	0.198	0.220	0.256	0.311	0.138	0.151	0.176	0.211
	$MBias$	0.450	0.512	0.602	0.774	0.301	0.335	0.388	0.375	0.204	0.232	0.267	0.319	0.143	0.159	0.184	0.229
IMH	$\overline{\alpha}$	2.065	2.113	2.142	2.109	2.027	2.050	2.077	2.060	2.019	2.031	2.031	2.021	2.012	2.026	2.019	2.007
	$\overline{\beta}$	1.947	1.948	1.970	2.035	1.977	1.972	1.991	2.015	1.996	1.995	1.996	2.002	1.997	1.995	1.995	2.004
	$RMSE$	0.416	0.462	0.567	0.723	0.279	0.308	0.360	0.454	0.198	0.222	0.245	0.306	0.140	0.155	0.176	0.210
	$MBias$	0.444	0.487	0.577	0.722	0.303	0.333	0.388	0.475	0.209	0.237	0.265	0.329	0.155	0.168	0.187	0.234
	$\%acc$	2.161	2.534	3.201	4.569	1.524	1.786	2.255	3.112	1.074	1.256	1.598	2.191	0.760	8.891	1.125	1.568
RWM	$\overline{\alpha}$	2.115	2.110	2.129	2.127	2.039	2.044	2.050	2.070	2.037	2.015	2.021	2.024	2.009	2.011	2.021	2.021
	$\overline{\beta}$	1.963	1.934	1.962	2.019	1.979	1.979	1.978	1.991	1.991	1.983	1.991	1.006	1.994	1.996	1.996	2.011
	$RMSE$	**0.408**	**0.455**	**0.548**	**0.677**	**0.277**	**0.305**	**0.358**	**0.442**	**0.195**	**0.206**	**0.242**	**0.305**	**0.132**	**0.148**	**0.174**	**0.209**
	$MBias$	**0.435**	**0.471**	**0.551**	**0.676**	**0.298**	**0.329**	**0.381**	**0.463**	**0.202**	**0.223**	**0.262**	**0.312**	**0.141**	**0.157**	**0.180**	**0.227**
	$\%acc$	22.756	25.597	32.471	41.703	16.112	18.900	23.496	31.318	11.492	13.455	16.884	22.779	8.126	9.513	12.037	16.490
SS	$\overline{\alpha}$	2.128	2.107	2.151	2.140	2.068	2.077	2.074	2.079	2.065	2.053	2.048	2.045	2.054	2.051	2.046	2.042
	$\overline{\beta}$	1.956	1.955	1.976	2.015	2.022	2.024	1.991	2.002	2.005	1.987	1.991	1.993	2.005	1.999	1.997	2.002
	$RMSE$	0.427	0.456	0.549	0.684	0.279	0.314	0.360	0.446	0.210	0.222	0.247	0.310	0.148	0.160	0.184	0.214
	$MBias$	0.446	0.473	0.559	0.696	0.391	0.451	0.383	0.469	0.231	0.236	0.268	0.318	0.161	0.175	0.199	0.235

8. Appendix B: Average of ESS and IAT

This section present the average values of ESS and IAT. Tables 11, 12 and 13 show the average values of ESS and IAT for $\alpha = 0.5, 1, 2$, respectively, for $\beta = 1$. As discussed in the text, SS presented better performance than IMH and RWM, for all simulated cases.

Table 11. Average of ESS and IAT by method. True parameters are $\alpha = 0.5$ and $\beta = 1$.

Sample size	% cens.	ESS						IAT					
		IMH		RWM		SS		IMH		RWM		SS	
		α	β	α	β	α	β	α	β	α	β	α	β
$n = 25$	0%	10.914	795.241	1583.969	2725.350	3007.365	3007.511	235.005	2.0573	1.983	1.149	1.0194	1.0217
	10%	13.271	2003.655	1894.993	2969.001	3009.271	3014.959	188.050	1.3012	1.635	1.040	1.0217	1.0211
	20%	14.163	2378.184	2033.750	2994.887	3013.765	3014.426	176.833	1.1726	1.530	1.032	1.0207	1.0225
	30%	14.770	1881.143	2071.532	2974.020	3013.457	3013.887	170.213	1.3297	1.505	1.041	1.0229	1.0230
$n = 50$	0%	10.454	662.594	1151.383	2617.865	3003.974	3019.616	274.418	2.2372	2.679	1.200	1.0253	1.0194
	10%	11.715	2054.291	1468.025	2957.595	3025.986	3012.777	231.395	1.2751	2.112	1.044	1.0207	1.0229
	20%	12.796	2560.670	1586.683	2995.093	3002.794	3014.110	213.372	1.1214	1.952	1.031	1.0223	1.0223
	30%	12.526	1917.031	1641.065	2965.575	3018.973	3006.601	218.281	1.2673	1.899	1.046	1.0208	1.0205
$n = 100$	0%	11.028	527.134	840.572	2462.283	3010.309	3010.545	311.021	2.7713	3.650	1.272	1.0253	1.0251
	10%	12.443	1989.185	1084.941	2957.989	3009.049	3018.396	266.081	1.3451	2.845	1.044	1.0241	1.0230
	20%	11.933	2520.803	1195.004	3003.145	3017.355	3009.566	254.151	1.1474	2.579	1.027	1.0231	1.0208
	30%	11.603	1761.648	1222.506	2949.266	3007.586	3011.072	253.337	1.3425	2.531	1.054	1.0239	1.0216
$n = 200$	0%	11.794	353.798	592.434	2260.110	3013.155	3008.449	328.180	4.1854	5.161	1.378	1.0225	1.0227
	10%	11.895	1774.786	776.829	2947.484	3013.209	3000.633	292.235	1.5353	3.934	1.052	1.0224	1.0241
	20%	12.531	2407.582	859.086	3000.828	3006.693	3006.679	279.872	1.2021	3.573	1.027	1.0217	1.0233
	30%	12.629	1504.042	891.319	2920.584	3015.037	3019.086	280.305	1.4977	3.448	1.059	1.0242	1.0186

Table 12. Average of ESS and IAT by method. True parameters are $\alpha = 1$ and $\beta = 1$.

Sample size	% cens.	ESS						IAT					
		IMH		RWM		SS		IMH		RWM		SS	
		α	β	α	β	α	β	α	β	α	β	α	β
$n = 25$	0%	11.264	794.673	2419.736	2913.059	3012.627	3014.594	230.582	2.060	1.273	1.065	1.022	1.023
	10%	12.322	1281.479	2561.085	2974.554	3018.099	3001.146	208.196	1.656	1.202	1.040	1.019	1.023
	20%	12.911	1883.033	2652.948	2994.813	3023.087	3018.130	196.841	1.374	1.161	1.029	1.022	1.020
	30%	14.138	2153.375	2719.746	2998.420	3009.461	3008.820	178.096	1.239	1.137	1.029	1.022	1.024
$n = 50$	0%	10.503	647.447	1999.249	2860.893	3006.620	3011.349	271.660	2.290	1.542	1.086	1.023	1.024
	10%	11.251	1237.764	2168.585	2948.539	3013.023	3003.899	255.996	1.702	1.418	1.052	1.026	1.024
	20%	11.639	1962.131	2324.082	2996.799	3011.954	3006.697	236.576	1.323	1.319	1.031	1.024	1.025
	30%	12.068	2376.468	2444.467	3006.634	3018.068	3016.384	215.508	1.178	1.262	1.026	1.021	1.021
$n = 100$	0%	11.113	502.106	1561.409	2774.631	3012.561	3014.684	304.594	2.731	1.975	1.123	1.024	1.023
	10%	12.571	1015.858	1730.375	2918.989	3007.278	3003.994	285.890	1.968	1.784	1.067	1.025	1.024
	20%	11.512	1835.407	1903.172	2978.579	3012.023	3009.155	269.729	1.472	1.619	1.037	1.023	1.023
	30%	12.671	2282.744	2035.642	2990.367	3010.814	3018.307	243.258	1.258	1.515	1.030	1.021	1.022
$n = 200$	0%	11.819	389.953	1151.451	2632.049	3011.119	3013.204	334.912	4.671	2.666	1.183	1.023	1.022
	10%	11.585	841.685	1289.762	2861.248	3015.521	3013.596	321.082	3.229	2.370	1.090	1.023	1.022
	20%	13.528	1776.513	1457.243	2964.628	3015.412	3007.889	303.747	1.682	2.106	1.042	1.022	1.025
	30%	13.770	2175.792	1579.200	3016.439	3009.794	3010.989	275.250	1.331	1.950	1.025	1.021	1.024

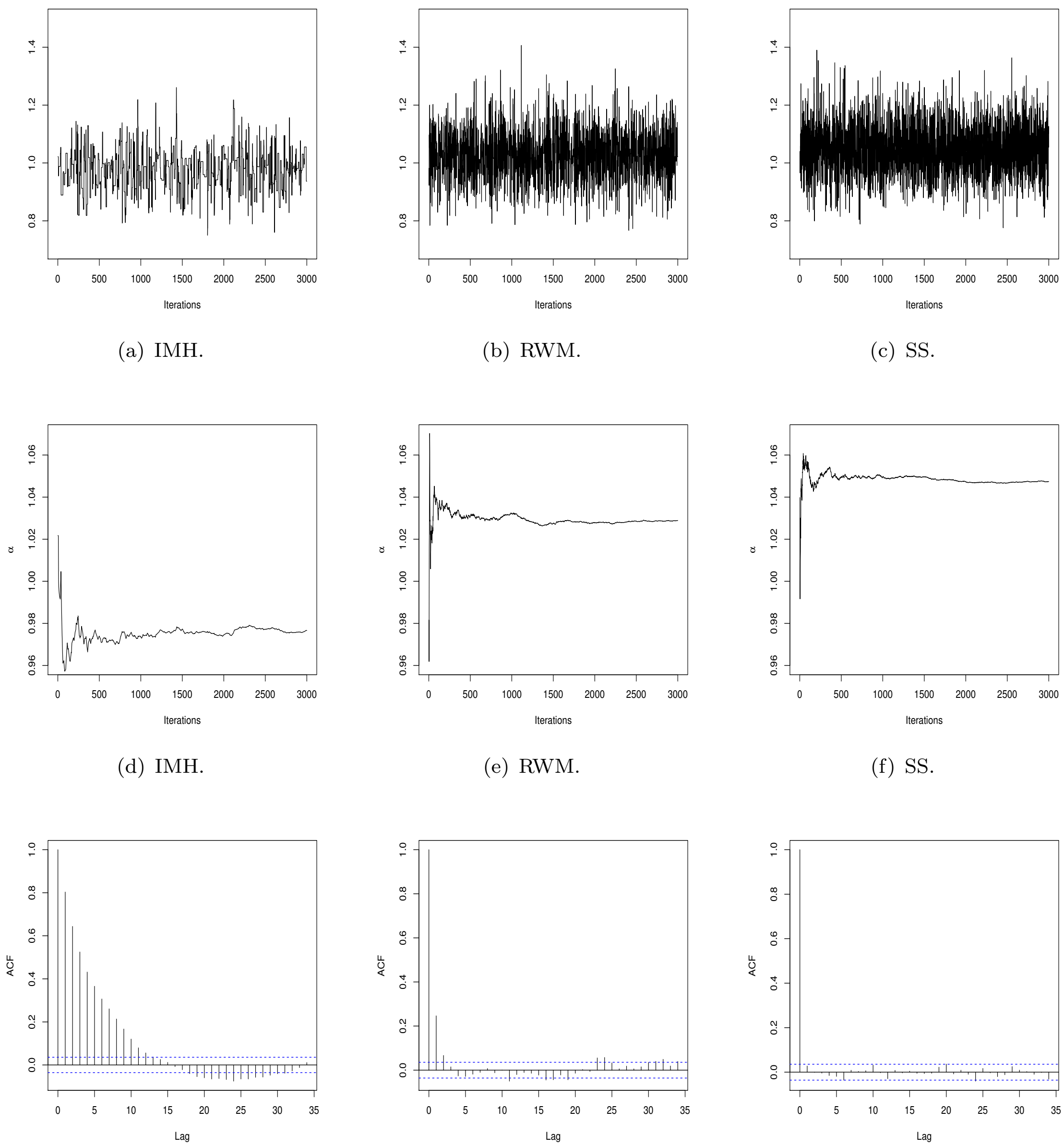
Table 13. Average of ESS and IAT by method. True parameters are $\alpha = 2$ and $\beta = 1$.

Sample size	% cens.	ESS						IAT					
		IMH		RWM		SS		IMH		RWM		SS	
		α	β	α	β	α	β	α	β	α	β	α	β
$n = 25$	0%	11.471	813.218	2873.160	2982.812	3015.625	3004.231	235.527	2.184	1.079	1.036	1.024	1.023
	10%	11.975	1114.657	2899.846	2994.253	3018.810	3000.619	214.906	1.856	1.069	1.028	1.021	1.024
	20%	12.749	1397.766	2928.011	3002.539	3010.388	3010.348	203.713	1.611	1.060	1.027	1.025	1.022
	30%	13.643	1678.263	2934.266	3000.876	3006.185	3008.167	187.757	1.450	1.058	1.026	1.024	1.022
$n = 50$	0%	10.448	630.884	2726.118	2966.378	3008.405	3015.781	274.573	2.330	1.133	1.041	1.025	1.024
	10%	10.848	909.238	2774.491	2993.956	3002.692	3018.706	257.625	1.992	1.115	1.031	1.023	1.021
	20%	11.245	1279.473	2834.220	2992.731	3007.845	3009.239	251.049	1.690	1.092	1.028	1.020	1.023
	30%	12.097	1767.193	2890.282	3008.806	3015.420	3018.861	231.344	1.413	1.075	1.026	1.024	1.024
$n = 100$	0%	12.254	481.225	2426.487	2945.122	3014.232	3006.232	309.072	3.111	1.264	1.055	1.022	1.025
	10%	11.489	698.791	2508.935	2968.794	3012.290	3012.697	299.728	2.421	1.225	1.044	1.022	1.023
	20%	11.829	1141.353	2599.839	2995.714	3013.949	3009.181	289.829	1.869	1.180	1.029	1.023	1.022
	30%	11.612	1728.197	2695.750	2999.057	3007.794	3011.513	273.264	1.528	1.143	1.027	1.021	1.023
$n = 200$	0%	13.146	359.136	1985.809	2875.204	3007.318	3014.285	337.372	5.656	1.538	1.080	1.021	1.024
	10%	11.529	580.206	2117.061	2927.680	3009.346	3004.812	339.057	3.890	1.444	1.058	1.024	1.021
	20%	12.371	906.850	2225.303	2968.866	3011.263	2998.029	301.780	2.787	1.372	1.043	1.022	1.025
	30%	15.907	1722.439	2344.668	2998.928	3010.315	3016.226	296.508	1.838	1.307	1.028	1.022	1.022

Apendix C: Empirical convergence

In this appendix we present an empirical illustration of the convergence of IMH, RWM and SS for the α sampled values. We select randomly a data set from one of the $M = 1,000$ generated and present the traceplot, the graphics of the ergodic mean and autocorrelation of the sampled values by method.

Figure 8 shows the traceplots, the ergodic mean and autocorrelation for sampled values for α. As we can note, sampled values by IMH does not mix well and stability for ergodic mean and autocorrelation are not satisfactory. In the other hand, sampled values by RWM and SS are well mixed and present satisfactory stability for ergodic mean and autocorrelation. For the β sampled values the three methods present satisfactory properties, *i.e.*, mix well and satisfactory stability for ergodic and autocorrelation.



(a) IMH. (b) RWM. (c) SS.

(d) IMH. (e) RWM. (f) SS.

(g) IMH. (h) RWM. (i) SS.

Figure 8. Traceplots, ergodic mean and autocorrelation by method.

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