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Times Series Models
Research Paper

GARCH-in-mean models with asymmetric variance processes for bivariate European option evaluation

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Abstract

Options pricing models, which consider asset-objects following a geometric Brownian motion, such as derivations from the traditional Black-Scholes model, assume the volatility of asset-objects to be constant over time. In addition, the normal distribution is the basement of the joint distribution for the case of bivariate options. In this work, we consider GARCH-in-mean models with asymmetric variance specifications to model the volatility of the assets-objects under the risk-neutral dynamics. Moreover, the copula functions model the joint distribution, with the objective of capturing non-linear, linear and tails associations between the assets. We provide a methodology to describe a more realistic pricing option. To illustrate the methodology, we use stocks from two Brazilian companies. Confronting the results obtained with the classic model, which is an extension of the Black-Scholes model, we note that considering constant volatility over time underpricing the options, especially in-the-money options. Overall, the contributions of the proposed methodology are as follows. Using the best copula makes the model more suitable. Extension to marginal models, which consider asymmetry, makes joint modeling more flexible and realistic. Due to the adequate marginal and joint fitting, in addition to the values obtained with the classical consolidated model, there are arguments to believe that the differences obtained between the best models, through the copulas and the extension of the conventional method, are improvements in the calculation of the fair value. The empirical relevance of such alternatives is apparent given the evidence of non-joint-normality in financial emerging markets. In essence, the entire approach may be generalized to any number of time-series of option pricing.

Keywords: Black-Scholes model · Copulas · GARCH models · Pricing.

Mathematics Subject Classification: Primary 62J99 · Secondary 62M20.

1. Introduction

Multivariate options are excellent tools to manage a portfolio's risk. The first works that had as objective the pricing of options in the univariate case were Black and Scholes (1973) and Merton (1973). Through these works, other authors have used the same theory, that is, asset-objects follow a Brownian geometric motion and have proposed bi and multivariate models, such as Stulz (1982), Margrabe (1978), Johnson (1987), Nelsen (2006) and

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Shimko (1994). However, models derived from Brownian geometric motion methods have the assumptions that the volatilities of the assets are constant over time.

To carry out the pricing with more realistic assumptions, researchers have developed other models. For instance, we use the generalized autoregressive conditional heteroskedasticity (GARCH) family of models, because of its ability to incorporate the stylized facts about asset return dynamics. This kind of modeling is popular in economics and finance (Almeida e Hotta, 2014). Furthermore, with Black-Scholes (BS) model assumptions, any contingent claim can be perfectly replicated by its underlying asset and a riskless bond, so the price of a contingent claim is merely the cost of the replicating portfolio. However, using GARCH-type models, it is generally not possible to construct a perfect replicating portfolio, as the volatility of asset returns is permitted to vary over time. It is necessary to define a risk-neutral measure to use the GARCH-type models to consider a general market equilibrium (Liu, Li and Ng, 2015).

The model of Duan (1995) derived a measure of risk-neutral through the standard GARCH model, which the author showed the potential of it concerning the Black-Scholes approach. However, one of the main limitations of the standard GARCH model is the inability to incorporate the effect of asymmetry caused by unplanned returns (Nelsen, 1991). Introduced by Black (1976), this effect implies that volatility tends to grow more when there is an unanticipated drop in returns (that is, bad news) than when there is an unanticipated increase of the same magnitude in returns (that is, good news). This effect, also known as a leverage effect, has been included in the GARCH-type models, such as the exponential GARCH (EGARCH), the non-linear asymmetric GARCH (NGARCH) and the Glosten, Jagannathan, and Runkle GARCH (GJR-GARCH) models. It can be used to price options by deriving their risk-neutral measure.

Furthermore, to understand the price behavior of a multivariate option, it is necessary to use tools that accommodate the co-movements between its underlying processes. A primary tool that is widely used by the methods derived from the traditional Black-Scholes model is the multivariate normal distribution modeling. However, the use of such an approach implies in linear associations as a measure of dependence between the assets. However, empirical evidence presents that a real association between financial series is much more complex (Lopes and Pessanha, 2018).

Therefore, this paper aims to price bivariate options by overcoming two of the above constraints of the classical approach, where asset-objects are modeled marginally by deriving their risk-neutral considering the GARCH, EGARCH, NGARCH and GJR-GARCH models, with copula functions modeling the joint distribution models, with the objective of capturing linear, non-linear and tails dependence. The entire methodology described here may be extended to any multivariate case.

An innovative feature of the present work is the comparison among methodologies, where we consider marginal processes that capture the effect of asymmetry, usually present in financial series. A second point is the performance of a simulation study of the pricing models with the purpose of verifying the good fit of the models used in the literature. It is highlighted as a third point the comparison of the methodology exposed to the standard method, extended from the Black-Scholes model to the bivariate case. The implementation of such methods in the Brazilian stock market, which is characterized as a volatile and unstable market concerning developed markets. Then, compared with the previous papers, the approach in the present paper makes the dynamic pricing more reasonable and tractable. The paper organization follows. Section 2 presents the conceptual framework and the models. In Section 3, we provide the bivariate model methodology and the inference procedures. In Section 4, the results of the proposed method under an artificial and real data sets are illustrated. Finally, Section 5 ends the paper with concluding remarks. Some technical details about different copulas are presented in the appendix.

2. Conceptual Framework and Model Specification

In this section, we present option pricing, the GARCH-in-mean specification and risk-neutral with GARCH-in-mean process.

2.1 Option pricing

A European option call on the maximum of two risky assets (call-on-max) is defined based on the maximum price between two assets. The payoff function of this option is given by

$$g(S(T)) = \max[\max(S_1(T), S_2(T)) - K, 0],$$

where S_i is the price of the ith asset, for $i = 1, 2$, at the maturity date T and K is the strike price or exercise price.

To introduce heteroscedasticity, we use the fundamental theorem of asset pricing (Delbaen and Schachermayer, 1994). This theorem states that once the stock prices $S_1(T)$ and $S_2(T)$ are free from arbitrage and present in a complete market (Hull, 1992), there is a measure of probability $\mathbb{Q}$ such that the discounted price of the payoff function, $\mathrm{e}^{-r(T-t)}g(S_1(T), S_2(T))$, is a martingale under $\mathbb{Q}$ and $\mathbb{Q}$ is equivalent to the real world probability measure $\mathbb{P}$. Therefore, we define the following definition to perform the pricing.

Definition 1. Let S_1 and S_2 be two stocks traded in a complete and free arbitrary market. In addition, be t the present date, T the maturity date and r the fixed risk-free rate yield. Then, the option price considering the payoff function $g(S_1, S_2) = \max[\max(S_1(T), S_2(T)) - K, 0]$ is given by

$$\begin{aligned} v(t, S_1, S_2) &= \mathrm{e}^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}[\max[\max(S_1(T), S_2(T)) - K, 0]|F_t] \\ &= \mathrm{e}^{-r(T-t)}\int_0^\infty\int_0^\infty \max[\max(S_1(T), S_2(T)) - K, 0] f^{\mathbb{Q}}_{S_1,S_2}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2, \end{aligned}$$

where $f^{\mathbb{Q}}_{S_1,S_2}$ is the joint density function of two measures under neutral risk probability $\mathbb{Q}$, which in this work is modeled by copula functions, and F_t is a filtering containing all information about the assets up to time t.

Now, we express the joint density function using the marginal densities $f_{S_1}(x_1)$ and $f_{S_2}(x_2)$ by means of copula functions expressed as

$$f^{\mathbb{Q}}_{S_1,S_2} = c^{\mathbb{Q}}(F^{\mathbb{Q}}_{S_1}, F^{\mathbb{Q}}_{S_2}) f^{\mathbb{Q}}_{S_1}(x_1) f^{\mathbb{Q}}_{S_2}(x_2),$$

where $c^{\mathbb{Q}} = \partial^2 C^{\mathbb{Q}}(x_1, x_2)/\partial x_1 \partial x_2$ and $C^{\mathbb{Q}}$ is a copula function.

Copulas are useful tools for constructing joint distributions (Sharifonnasabi, Alamatsaz and Kazemi, 2018). That is, copula is a multidimensional distribution function in which the marginal distributions are uniform in $[0, 1]$. A bivariate copula is a function $C : I^2 \longrightarrow I \in [0, 1]$ that satisfies the following conditions: $C(x_1, 0) = C(0, x_1) = 0 \quad \textit{and} \quad C(x_1, 1) = C(1, x_1) = x_1, \quad x_1 \in I$ and the 2-increasing condition $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0$, for all u_1, u_2, v_1 and $v_2 \in [0, 1]$ such $u_1 \leq u_2$ and $v_1 \leq v_2$.

One of the most famous theorems in copula theory is the Sklar theorem. According to Sklar's theorem (Sklar, 1959), any bivariate cumulative distribution H_{S_1,S_2} can be represented as a function of the marginal distributions F_{S_1} and F_{S_2}. In addition, whether the marginal distributions are continuous, the copula exists, is unique and given by

$$H_{S_1,S_2}(x_1,x_2) = C(F_{S_1}(x_1), F_{S_2}(x_2)),$$

where $C(u,v) = P(U \leq u, V \leq v)$, $U = F_{S_1}(x_1)$ and $V = F_{S_2}(x_2)$.

In the case of continuous and differentiable marginal distributions, the joint density function of the copula is expressed as

$$f(x_1,x_2) = f_{S_1}(x_1) f_{S_2}(x_2) c(F_{S_1}(x_1), F_{S_2}(x_2)),$$

where $f_{S_1}(x_1)$ and $f_{S_2}(x_2)$ are the densities for the distribution function $F_{S_1}(x_1)$ and $F_{S_2}(x_2)$, respectively, and

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial uv}$$

is the density of copula. For further details about copulas, see Nelsen (2006) and Sanfins and Valle (2012). In this work, we consider the normal, Student-t, Gumbel, Frank and Joe copulas. An appendix at the end of this paper provides details about these copula functions. Therefore, to construct a joint process of risk-neutral for the bivariate distribution of the option, the marginal processes are derived first.

2.2 GARCH-in-mean specification under $\mathbb{P}$

Instead of deriving the bivariate risk-neutral distribution directly, each marginal process is proposed to transform separately. Duan (1995) defined an option pricing model considering that the variance of the asset-object is not constant over time. To implement non-constant volatility over the maturity time of the option, we use in this work the generalized autoregressive conditional heteroskedastic (GARCH) models. Bollerslev (1986) introduced the GARCH model by modifying the ARCH model presented by Engle (1982). The use of GARCH models in pricing leads to the correction of some biases in the model of Black and Scholes (1973), including return skewness and leptokurtic behavior.

GARCH-in-mean refers to the inclusion of an extra term m_t in the conditional mean of the model introduced by Bollerslev (1986). An intuitive idea to use these models in derivative pricing is that conditional variance is not constant over time and hence the conditional mean of market returns is a linear function of conditional variance. Another definite reason to work with the GARCH-in-mean models is that these models explain the presence of conditional left skewness observed in stock returns.

Consider a discrete time economy with a risk-free asset. We define a complete filtered probability space $(\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$ to model uncertainty, where $\mathbb{P}$ is the historical (physical) measure and $\mathbb{F} = \mathbb{F}_t$, for $t = 0, 1, \ldots, T$, is a filtration, or a family of increasing σ-field information sets, representing the resolution of uncertainty based on the information generated by the market prices up to and including time t. We assume the general GARCH-M(p,q) model for the return $y_t = \log(S_t/S_{t-1})$ given by

$$y_t = m_t + \sqrt{h_t}\epsilon_t, \quad h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i h_{t-i}\phi(\epsilon_{t-i}) + \sum_{i=1}^{q} \beta_i h_{t-i}, \tag{1}$$

where S_t is the stock price at time t and ϵ_t is a sequence of independent and identically distributed random variables with normal distribution; the conditional mean return m_t is assumed to be an F_t-predictable process. In many studies, m_t is assumed to be a function of the conditional variance h_t of the return and a risk premium quantifier at time t; the function ϕ describes the impact of random shock of return ϵ_t on the conditional variance h_t and $\alpha_0 > 0, \alpha_i$ and $\beta_i \geq 0$.

The conditional mean and variance of y_t are $m_t = \mathrm{E}[y_t|F_{t-1}]$ and $h_t = \mathrm{Var}[y_t|F_{t-1}]$. The effect of past innovations ϵ_{t-1} under the conditional variance h_t have different impacts depending on the function $\phi(\epsilon_{t-1})$, and consequently we have different extensions of the GARCH model. For example, considering $p = q = 1$, when $\phi(\epsilon_{t-1}) = \epsilon_{t-1}^2$, the sign of ϵ_{t-1} there is no effect over h_t, and we have the traditional GARCH proposed by Bollerslev (1986). Thus, the innovations have a symmetric effect on the conditional variance, expressed by

$$h_t = \alpha_0 + \alpha_1 h_{t-1}\epsilon_{t-1}^2 + \beta_1 h_{t-1}. \tag{2}$$

Following Liu, Li and Ng (2015), Duan (1995) and Chiou and Tsay (2008), $m_t = r + \lambda\sqrt{h_t} - k_{\epsilon_t}(\sqrt{h_t})$, where $k_{\epsilon_t}(\sqrt{h_t})$ is the cumulate generating function of the innovation ϵ_t e λ is the premium risk parameter. When ϵ_t follows a normal distribution, we have $k_{\epsilon_t}(\sqrt{h_t}) = h_t/2$. Because standard GARCH models given by equation (1) respond in the same way to positive and adverse events, such models cannot correctly capture the leverage effect. Other forms of the GARCH model, such as EGARCH, NGARCH, and GJR-GARCH, include the asymmetry effect, can thus be used in option pricing and are used in the present work. Nelsen (1991) proposed the exponential GARCH (EGARCH) model. The author assumes that the dynamic of the logarithm of the conditional variance of EGARCH(1,1) is expressed as

$$\log(h_t) = \alpha_0 + \alpha_1(|\epsilon_{t-1}| + \gamma_1\epsilon_{t-1}) + \beta_1 \log(h_{t-1}), \tag{3}$$

where α_0, α_1, β_1 and γ_1 are constant parameters and ϵ forms a sequence of independent standard normal random variables representing random shocks. The EGARCH model does not require such parameter restrictions since the conditional variance is expressed as the exponential of a function. Including the random shock term in absolute value and with a parameter γ_1, the author made volatility a function of both magnitude and sign of the shock.

Engle (1982) introduced the non-linear asymmetric GARCH (NGARCH), which takes into account the leverage effect. In their model, the dynamic of the conditional variance of NGARCH(1,1) is given by

$$h_t = \alpha_0 + \alpha_1 h_{t-1}(\epsilon_{t-1} - \gamma_1)^2 + \beta_1 h_{t-1}, \tag{4}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and γ_1 is a non-negative parameter that captures the negative correlation between return and volatility innovations. Since the parameter α_1 is typically non-negative, a positive γ_1 means that negative random shocks increase volatility more than positive random shockes of similar magnitude. Hence, the NGARCH allows for the levarage through its paramater γ_1.

Another model that takes into account the asymmetry effect of news on volatility is the GJR-GARCH introduced by Glosten, Jagannathan and Runkle (1993). According to this model, the conditional variance dynamic of GJR-GARCH(1,1) is defined as

$$h_t = \alpha_0 + \alpha_1 h_{t-1}\epsilon_{t-1}^2 + \beta_1 h_{t-1} + \gamma_1 h_{t-1} \max(0, -\epsilon_{t-1})^2, \tag{5}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\gamma_1 \geq 0$ are constant parameters. This model allows for the leverage effect by adding the extra term $\gamma_1 h_{t-1} \max(0, -\epsilon_{t-1})^2$ when ϵ_t is negative since γ_1 is typically non-negative.

All the models presented above are in the physical measure ($\mathbb{P}$ measure). Now, we discuss their representations in the risk-neutral measure ($\mathbb{Q}$ measure), a prerequisite for pricing options under heteroscedasticity.

2.3 Risk-neutral with GARCH-in-mean process

The concept of risk-neutral valuation relationship (RNVR) has a fundamental role in the process of pricing options. This principle has as the base an asset, which is priced according to the discount of the expected value of a payoff function under a martingale measure, that is, that the economic agents are risk-neutral.

To apply this pricing methodology, we assume that a measure of martingale $\mathbb{Q}$ exists in a discrete economy time, with interest rate and a probability space $(\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$, where $\mathbb{P}$ is a measure of physical probability and $\mathbb{F}_t$ is a filtering at time t.

Definition 2. A measure of probability $\mathbb{Q}$ is equivalent to a measure of probability $\mathbb{P}$ if:

(1) $\mathbb{Q} \approx \mathbb{P}$, that is, for all event X, $\mathbb{Q}(X) = 0$ and $\mathbb{P}(X) = 0$.

(2) The discounted price process S_t is a martingale under $\mathbb{Q}$, that is, $\mathrm{E}^{\mathbb{Q}}[S_t|F_{t-1}] = S_{t-1}$.

Proposition. Assuming continuously compounded returns, the martingale condition for the discounted stock price can be replaced by

$$\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{y_t}|F_{t-1}] = \mathrm{e}^r.$$

Proof. From second condition in Definition 2, we have

$$\mathrm{E}^{\mathbb{Q}}[S_t|F_{t-1}] = S_{t-1} \Leftrightarrow \mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-rT} S_t|F_{t-1}] = \mathrm{e}^{r(t-1)} S_{t-1} \Leftrightarrow \mathrm{E}^{\mathbb{Q}}\left[\frac{S_t}{S_{t-1}}|F_{t-1}\right] = \mathrm{e}^r$$
$$\Leftrightarrow \mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{y_t|F_{T-1}}] = \mathrm{e}^r.$$

Brennan and Schwartz (1979) represented a starting point by providing conditions which ensure the existence of the risk-neutral measure. Duan (1995) proposes an extension of RNVR, referred to as locally risk-neutral valuation relationship (LRNVR) by assuming a conditional Gaussian distribution for the log-returns with unchanged volatility after the change of measure.

Definition 3. A no arbitrage measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ is said to satisfy the local risk-neutral valuation relationship (LRNVR) if the following conditions are satisfied:

(1) $y_t|F_{t-1} \sim \mathrm{N}(m_t, h_t)$ under $\mathbb{P}$, where $\epsilon_t \sim \mathrm{N}(0, 1)$.

(2) $\mathrm{E}^{\mathbb{Q}}[S_t/S_{t-1}|F_{t-1}] = \mathrm{e}^r$.

(3) $\mathrm{Var}^{\mathbb{Q}}[\log(S_t/S_{t-1})|F_{t-1}] = \mathrm{Var}^{\mathbb{P}}[\log(S_t/S_{t-1})|F_{t-1}]$.

In the previous definition, the conditional variance under the two measures is required to be equal. This requirement is necessary to estimate the conditional variance under $\mathbb{P}$ and use the framework to obtain the option pricing under $\mathbb{Q}$. This property and the fact of the risk-free rate can replace the conditional mean, yield a well-specified model that does not locally depend on preferences. Duan (1995) proved this latter fact. Here we reduce all preference consideration to the unit risk premium λ. Since $\mathbb{Q}$ is absolutely continuous for $\mathbb{P}$, the almost certain relationship under $\mathbb{P}$ also holds true under $\mathbb{Q}$. Duan (1995) and Duan et al. (2006) showed that under the risk-neutral measure $\mathbb{Q}$ given by LRNVR, the asset return dynamic becomes

$$y_t = r - \frac{1}{2}h_t + \sqrt{h_t}\tilde{\epsilon}_t, \quad \tilde{\epsilon}_t \sim \mathrm{N}(0,1).$$

In addition:
GARCH(1,1): $h_t = \alpha_0 + \alpha_1 h_{t-1}(\tilde{\epsilon}_{t-1} - \lambda_1)^2 + \beta_1 h_{t-1}$.
EGARCH(1,1): $h_t = \alpha_0 + \alpha_1[|\tilde{\epsilon}_{t-1} - \lambda_1| + \gamma_1(\tilde{\epsilon}_{t-1} - \lambda_1)] + \beta_1 \log(h_{t-1})$.
NGARCH(1,1): $h_t = \alpha_0 + \alpha_1 h_{t-1}(\tilde{\epsilon}_{t-1} - \gamma_1 - \lambda_1)^2 + \beta_1 h_{t-1}$.
GJR-GARCH(1,1): $h_t = \alpha_0 + h_{t-1}[\beta_1 + \alpha_1(\tilde{\epsilon}_{t-1} - \lambda_1)^2 + \gamma_1 \max(0, -\tilde{\epsilon}_{t-1} + \lambda_1)^2]$.
Under LRNVR, the form of m_t just affects the volatility dynamics while the risk-neutralized conditional mean return remains the same, that is, $r - h_t/2$. Now, we have all the variance specification in the risk-neutral measure. According to the equations above, the final asset price is derived from Corollary 1.

Corollary 1. When the locally risk-neutral valuation relationship holds, the terminal price for the ith asset, for $i = 1, 2$, can be expressed as

$$S_{i,T} = S_{i,t}\,\mathrm{e}^{(T-t)r} - \frac{1}{2}\sum_{s=t+1}^{T} h_{i,s} + \sum_{s=t+1}^{T} \sqrt{h_{i,s}}\tilde{\epsilon}_{i,s}].$$

Therefore, under the locally risk-neutral probability measure $\mathbb{Q}$, the option with exercise price K at maturity T has the value

$$v(t, S_1, S_2) = \mathrm{e}^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}[\max[\max(S_1(T), S_2(T)) - K, 0]].$$

Due to the complexity of the GARCH process, analytical solution for the GARCH-in-mean Copula option-pricing model, in general, is not available. Therefore, we work with numerical methods to price the option described in the next section.

3. Methodology and Inference

In this section, we present here the procedure to obtain the price of a bivariate option using the asymmetric variance process by GARCH-in-mean under risk-neutral, copulas theory and Monte Carlo simulations. Chiou and Tsay (2008) and Zhang and Guegan (2008) have inspired this approach.

3.1 Generality

Given y_1 and y_2, two vectors containing the log-returns for the two stocks, we consider the following steps:

(1) For each y_i, with $i = 1, 2$, use quasi-maximum likelihood method described in Subsection 3.2 to estimates the parameters α_0, α_1, β_1 and λ in equation (2) and α_0, α_1, β_1, γ and λ for each marginals given in equations (3), (4) and (5). Thus, the problem is to maximize the function

$$l(\boldsymbol{\theta}, h_t) = -\frac{n}{2}\left[\log(2\pi) + \frac{1}{n}\sum_{t=1}^{n}\left[\log(h_t) + \frac{(y_{it} - m_{it})^2}{h_t}\right]\right],$$

with respect to the parameters, where m_{it} is the mean of GARCH-in-mean given by $r + \lambda\sqrt{h_t} - 1/2h_t$ and r is the fixed risk-free rate yield and h_t corresponds to each variance specification proposed in Subsection 2.2.

(2) Use the estimated parameters to calculate h_t for each specification and ϵ_t in equation (1) with $m_t = r + \lambda\sqrt{h_t} - 1/2h_t$ for each stock.

(3) Therefore, the proposed technique is that the objective copula and the risk-neutral copula are assumed to be the same. To fit the copulas, we transform the data into uniformly distributed random variables. Thus we transform the ϵ_i, for $i = 1, 2$, obtained in Step 2 for each stock into uniformly distributed variables, by $u_i = \Phi(\epsilon_i)$, where Φ is the standard normal cumulative distribution function.

(4) Fit a copula to pairs $[u_1, u_2]$ using maximum likelihood, that is, estimate the copula parameters θ_c

$$\theta_c = \arg\max_{\theta_c} \sum_{t=1}^{n} \log[c((u_{1,t}, u_{2,t}); \theta_c)],$$

where θ_c are the parameters for the specific copula function C and c is the density function for the given copula in the appendix.

(5) Now, using the Monte Carlo simulation, we obtain the option price. In the first step generate a sample $\{u^*_{1,t}, u^*_{2,t}\}_{t=1}^{T}$ from a uniform marginal distribution from one specific copula using the algorithm proposed by Nelsen (2006). Here T is the time to maturity for the option.

(6) For each time step, transform the generated margins to standard normal margins, in the risk-neutral measure, by $\tilde{\epsilon}_{i,t} = \Phi^{-1}(u^*_{i,t})$, for $i = 1, 2$.

(7) Working with $\tilde{\epsilon}_{i,t}$ to calculate the conditional variances under risk-neutral and the parameters estimated in step 1. The two future stock prices at time T are

$$S_{i,T} = S_{i,t}\, \mathrm{e}^{(T-t)r} - \frac{1}{2}\sum_{s=t+1}^{T} h_{i,s} + \sum_{s=t+1}^{T} \sqrt{h_{i,s}}\tilde{\epsilon}_{i,s}].$$

(8) Now, repeat Steps 5 to 7 for N runs. Thus, we obtain the Monte Carlo option price as

$$v(t, S_1, S_2) = \frac{\mathrm{e}^{-r(T-t)}}{N} \sum_{i=1}^{N} \max[\max(S_{1,i}(T), S_{2,i}(T)) - K, 0].$$

3.2 Quasi-maximum likelihood estimation

The assumption of conditional normality is not always appropriate in financial data. However, Weiss (1986) and Bollerslev and Wooldridge (1992) showed that even when normality is inappropriately assumed, maximizing the normalized log-likelihood results in quasi-maximum likelihood (QML) estimates that are consistent and asymptotically normally distributed. In addition, the authors claim that the conditional mean and variance functions of the GARCH models are correctly specified.

In particular, a robust covariance matrix conditional non-normality for the parameter estimates is consistently estimated by $A(\hat{\boldsymbol{\theta}})^{-1}B(\hat{\boldsymbol{\theta}})A(\hat{\boldsymbol{\theta}})^{-1}$, where $A(\hat{\boldsymbol{\theta}})$ and $B(\hat{\boldsymbol{\theta}})$ are the Hessian Matrix and the outer product of the gradients, respectively, calculated for $\boldsymbol{\theta}$. The SEs, computed from the square roots of the diagonal elements, are sometimes called Bollerslev-Wooldridge SE; for more details, see Bollerslev and Wooldridge (1992).

3.3 Model selection

We notice that for each time series we have four specification for variance processes, that is, GARCH(1,1), EGARCH(1,1), NGARCH(1,1) and GJR-GARCH(1,1). Choosing an adequate model is the essence of data analysis, which ultimately returns with good forecasting results.

In this paper, for model selection, we use five different criteria. The first one is the Akaike information criterion (AIC) (Akaike, 1973) given by AIC $= -2\log(\ell) + 2k$, where ℓ is the maximized value of the likelihood function and k is the number of free parameters in the model. The second one is the Bayesian information criterion (BIC) developed by Schwarz (1978) and given by BIC $= -2\log(\ell) + k\log(n)$, where n is the number of observations. The third one is the Hannan-Quinn information criterion (HQIC) proposed by Hannan and Quinn (1979) and given by HQIC $= -2\log(\ell) + 2k\log(\log(n))$. The fourth one is the Akaike information corrected criterion (AICc), developed by Hurvich and Tsai (1989) and given by AICc $= -2\log(\ell) + 2kn/(n-k-1)$, whereas the fifth one is the consistent Akaike information criterion (CAIC) given by $-2\log(\ell) + k\log(n) + 1$.

Following Genest, Remillard and Beaudoin (2009), we use the goodness-of-fit test, which is based on a comparison of the distance between the estimated and empirical copula by using the Cramer Von Mises test to compare the copula models. The goodness-of-fit statistic is defined as

$$S_n = \int_{[0,1]^d} \mathbb{C}_n(\boldsymbol{u})^2 \mathrm{d}C_n(\boldsymbol{U}),$$

where $C_n(\boldsymbol{U}) = 1/n\sum_{i=1}^n \mathbb{I}(U_{i1} \le u_1; U_{i2} \le u_2)$ is known as the empirical copula; $\boldsymbol{U}_j = (U_{1j}, \ldots, U_{ij})$ are the pseudo-observations; $\boldsymbol{u} = (u_1, u_2) \in [0,1]^2$; $\mathbb{C}_n = \sqrt{n}(C_n - C_{\theta_n})$ is the empirical process that assess the distance between the empirical copula and the estimation C_{θ_n} and n is the number of observations. Note that testing the null hypothesis that data are fitted by C_{θ_n} can be conducted with this statistic.

We chose this procedure because it can deal with non-linearity, asymmetry, serial dependence and also the well-known heavy-tails of financial assets (Righi and Ceretta, 2011). Furthermore, we make the comparison of the adjusted copula with the empirical copula by the diagonal method Sungur and Yang (1996). In addition, the AIC, AICc, CAIC, BIC and HQICare also used to support decision making in choosing the model.

4. Data Analyses

In this section, we illustrate the proposed methodology under two data sets. We used the software R for implementing the entire methods exposed here. The codes are available from the authors. The first one is artificial data, where we know the parameter values, and then we can verify if the methodology is reliable. The second data set is the Brazilian stock market data.

4.1 Artificial data

We consider here 1000 replications of two correlated time-series for each sample size ($n = 250, 500, 1000$) generated from same parameter structure with the Frank ($\theta = 8$) and marginals as follows:

GARCH(1,1):

$$h_{1,t} = 0.02 + 0.15h_{t-1}(\tilde{\epsilon}_{t-1} - 0.12)^2 + 0.8h_{t-1},$$
$$h_{2,t} = 0.03 + 0.2h_{t-1}(\tilde{\epsilon}_{t-1} - 0.08)^2 + 0.7h_{t-1},$$

EGARCH(1,1):

$$h_{1,t} = -0.3057 + 0.1223[|\tilde{\epsilon}_{t-1} - 0.12| + (-0.5057)(\tilde{\epsilon}_{t-1} - 0.12)] + 0.98ln(h_{t-1}),$$
$$h_{2,t} = -0.3057 + 0.1223[|\tilde{\epsilon}_{t-1} - 0.12| + (-0.5057)(\tilde{\epsilon}_{t-1} - 0.12)] + 0.98ln(h_{t-1}),$$

NGARCH(1,1):

$$h_{1,t} = 0.012 + 0.15h_{t-1}(\tilde{\epsilon}_{t-1} - 0.5 - 0.12)^2 + 0.8h_{t-1},$$
$$h_{2,t} = 0.03 + 0.2h_{t-1}(\tilde{\epsilon}_{t-1} - 0.2 - 0.08)^2 + 0.7h_{t-1},$$

GJR-GARCH(1,1):

$$h_{1,t} = 0.00961 + h_{t-1}[0.93 + 0.024(\tilde{\epsilon}_{t-1} - 0.065)^2 + 0.059max(0, -\tilde{\epsilon}_{t-1} + 0.065)^2],$$
$$h_{2,t} = 0.00961 + h_{t-1}[0.93 + 0.024(\tilde{\epsilon}_{t-1} - 0.065)^2 + 0.059max(0, -\tilde{\epsilon}_{t-1} + 0.065)^2].$$

For each configuration, we calculate the average of the QML estimates, as well as the corresponding robust standard error (SE) , the size of confidence intervals 95% (CI), coverage probability (CP), bias and mean squared error (MSE) of the QML estimators. Tables 1, 2, 3 and 4 report the simulation results for GARCH, NGARCH, EGARCH, and GJR-GARCH, respectively. We observe that the averages of the quasi-maximum likelihood estimates are close to the true values as the sample size increases, as well as decreasing the standard deviations in all the models. We also note low bias and MSEs as the sample size increases. Concerning the size of the confidence interval, we noticed they are getting smaller as the sample size increases. In addition, the empirical coverages are closer to the nominal ones for all four models. With this results, we noticed that all the models have good asymptotic properties.

Table 1. Parameter estimation of both artificial time-series for each GARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	θ
	Real Value	0.02	0.15	0.8	0.12	0.03	0.2	0.7	0.08	8
$n = 250$	Mean	0.0355	0.1479	0.7511	0.1192	0.0429	0.2020	0.6442	0.0834	7.9349
	SE	0.1695	0.4596	0.9783	0.1798	0.0337	0.0927	0.1983	0.0786	0.5777
	CI size	0.1213	0.2297	0.5095	0.2345	0.1347	0.2784	0.6594	0.1919	2.4122
	CP	0.9880	0.9490	0.9560	0.9760	0.9480	0.9289	0.9480	0.9750	0.9229
	Bias	-0.0155	0.0021	0.0489	0.0009	-0.0129	-0.0020	0.0558	-0.0034	0.0652
	MSE	0.0002	0.0000	0.0024	0.0000	0.0002	0.0000	0.0031	0.0000	0.0042
$n = 500$	Mean	0.0255	0.1492	0.7830	0.1198	0.0347	0.1998	0.6806	0.0825	7.9836
	SE	0.0126	0.0403	0.0615	0.0479	0.0156	0.0562	0.0917	0.0523	0.4092
	CI size	0.0493	0.1462	0.2403	0.1838	0.0541	0.1992	0.3349	0.1572	1.6726
	CP	0.9720	0.9500	0.9570	0.9470	0.9470	0.9269	0.9289	0.9720	0.9399
	Bias	-0.0055	0.0008	0.0170	0.0003	-0.0047	0.0002	0.0194	-0.0025	0.0164
	MSE	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0004	0.0000	0.0003
$n = 1000$	Mean	0.0223	0.1500	0.7931	0.1181	0.0323	0.2004	0.6908	0.0780	8.0054
	SE	0.0089	0.0314	0.0437	0.0403	0.0112	0.0413	0.0659	0.0401	0.2896
	CI size	0.0282	0.0990	0.1442	0.1251	0.0360	0.1308	0.1885	0.1111	1.2808
	CP	0.9600	0.9580	0.9570	0.9530	0.9439	0.9550	0.9550	0.9600	0.9469
	Bias	-0.0023	0.0000	0.0069	0.0019	-0.0023	-0.0004	0.0092	0.0020	-0.0054
	MSE	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000

Table 2. Parameter estimation of both artificial time-series for each NGARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	γ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	γ_2	θ
	Real value	0.012	0.15	0.8	0.12	0.5	0.03	0.2	0.7	0.08	0.2	8
$n = 250$	Mean	0.0266	0.1431	0.7656	0.1162	0.5517	0.0416	0.1962	0.6503	0.0798	0.3461	7.9343
	SE	0.1107	0.7069	0.7901	0.5111	6.3509	0.0444	0.1538	0.2345	0.1130	1.1610	0.5773
	CI size	0.0923	0.2061	0.3251	0.2349	0.9699	0.1046	0.2832	0.5296	0.1851	0.9900	2.5200
	CP	0.9860	0.9580	0.9620	0.9820	0.9730	0.9620	0.9429	0.9499	0.9870	0.9960	0.9289
	Bias	-0.0146	0.0069	0.0344	0.0038	-0.0517	-0.0116	0.0038	0.0497	0.0002	-0.0461	0.0657
	MSE	0.0002	0.0000	0.0012	0.0000	0.0027	0.0001	0.0000	0.0025	0.0000	0.0021	0.0043
$n = 500$	Mean	0.0162	0.1435	0.7894	0.1178	0.5345	0.0348	0.1972	0.6796	0.0811	0.3250	7.9601
	SE	0.0277	0.0744	0.1240	0.0964	0.4538	0.0149	0.0510	0.0778	0.0542	0.1790	0.4083
	CI size	0.0296	0.1412	0.1665	0.1859	0.8008	0.0553	0.2005	0.2963	0.1601	0.9699	1.7088
	CP	0.9860	0.9399	0.9520	0.9730	0.9620	0.9540	0.9299	0.9520	0.9740	0.9640	0.9269
	Bias	-0.0042	0.0065	0.0106	0.0022	-0.0345	-0.0048	0.0028	0.0204	-0.0011	-0.0250	0.0399
	MSE	0.0000	0.0000	0.0001	0.0000	0.0012	0.0000	0.0000	0.0004	0.0000	0.0006	0.0016
$n = 1000$	Mean	0.0142	0.1479	0.7934	0.1174	0.5167	0.0323	0.1973	0.6921	0.0789	0.3135	7.9780
	SE	0.0132	0.0425	0.0577	0.0535	0.1983	0.0121	0.0406	0.0625	0.0404	0.1285	0.2890
	CI size	0.0167	0.0971	0.1066	0.1322	0.4957	0.0336	0.1274	0.1810	0.1291	0.6677	1.1894
	CP	0.9520	0.9469	0.9600	0.9640	0.9590	0.9580	0.9479	0.9540	0.9640	0.9590	0.9479
	Bias	-0.0022	0.0021	0.0066	0.0026	-0.0167	-0.0023	0.0027	0.0079	0.0011	-0.0135	0.0220
	MSE	0.0000	0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0001	0.0000	0.0002	0.0005

Table 3. Parameter estimation of both artificial time-series for each EGARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	γ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	γ_2	θ
	Real Value	-0.3067	0.1223	0.98	0.12	-0.5057	-0.3067	0.1223	0.98	0.12	-0.5057	8
$n = 250$	Mean	-0.2852	0.0876	0.9793	0.1204	-0.5080	-0.2874	0.0891	0.9793	0.1204	-0.5065	7.9388
	SE	0.8405	1.1478	0.0244	0.2555	0.3996	0.6754	0.7820	0.0242	0.2362	0.2318	0.5781
	CI size	0.3182	0.2010	0.0180	0.2353	0.1900	0.3250	0.2215	0.0176	0.2367	0.1876	2.5276
	CP	0.9139	0.8759	0.8829	0.9570	0.9249	0.9149	0.8679	0.8749	0.9640	0.9269	0.9139
	Bias	-0.0205	0.0347	0.0007	-0.0004	0.0023	-0.0183	0.0332	0.0007	-0.0004	0.0008	0.0612
	MSE	0.0004	0.0012	0.0000	0.0000	0.0000	0.0003	0.0011	0.0000	0.0000	0.0000	0.0037
$n = 500$	Mean	-0.2960	0.1070	0.9798	0.1205	-0.5060	-0.2968	0.1082	0.9798	0.1198	-0.5067	7.9560
	SE	0.0649	0.0374	0.0035	0.0515	0.0408	0.0607	0.0493	0.0026	0.0571	0.0437	0.4085
	CI size	0.1790	0.1478	0.0081	0.1756	0.1239	0.1818	0.1592	0.0084	0.1707	0.1248	1.6859
	CP	0.9069	0.9118	0.9009	0.9289	0.9139	0.9179	0.9278	0.9309	0.9379	0.9199	0.9339
	Bias	-0.0097	0.0153	0.0002	-0.0005	0.0003	-0.0089	0.0141	0.0002	0.0002	0.0010	0.0440
	MSE	0.0001	0.0002	0.0000	0.0000	0.0000	0.0001	0.0002	0.0000	0.0000	0.0000	0.0019
$n = 1000$	Mean	-0.3033	0.1171	0.9799	0.1212	-0.5061	-0.3039	0.1176	0.9799	0.1210	-0.5054	7.9865
	SE	0.0391	0.0226	0.0020	0.0362	0.0242	0.0456	0.0233	0.0026	0.0457	0.0448	0.2892
	CI size	0.1116	0.0842	0.0047	0.1231	0.0733	0.1098	0.0838	0.0048	0.1286	0.0707	1.2940
	CP	0.9459	0.9409	0.9591	0.9429	0.9599	0.9449	0.9689	0.9339	0.9419	0.9489	0.9579
	Bias	-0.0024	0.0052	0.0001	-0.0012	0.0004	-0.0018	0.0047	0.0001	-0.0010	-0.0003	0.0135
	MSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002

Table 4. Parameter estimation of both artificial time-series for each GJR-GARCH process.

	Parameter	$\alpha_{0,1}$	$\alpha_{1,1}$	β_1	λ_1	γ_1	$\alpha_{0,2}$	$\alpha_{1,2}$	β_2	λ_2	γ_2	θ
	Real Value	0.00961	0.024	0.93	0.065	0.059	0.00961	0.024	0.93	0.065	0.059	8
$n = 250$	Mean	0.0582	0.0306	0.8326	0.0741	0.0548	0.0524	0.0334	0.8346	0.0725	0.0563	7.9335
	SE	3.1908	1.2611	7.8901	1.5276	1.5033	6.3504	7.0343	1.9827	6.7417	7.2567	0.5774
	CI size	0.4059	0.0949	0.9637	0.1953	0.1697	0.3703	0.1123	0.9627	0.1965	0.1697	2.5486
	CP	0.9970	0.9970	0.9880	0.9800	0.9990	0.9870	0.9990	0.9790	0.9840	0.9990	0.9199
	Bias	-0.0486	-0.0066	0.0974	-0.0091	0.0042	-0.0428	-0.0094	0.0954	-0.0075	0.0027	0.0665
	MSE	0.0024	0.0000	0.0095	0.0001	0.0000	0.0018	0.0001	0.0091	0.0001	0.0000	0.0044
$n = 500$	Mean	0.0219	0.0260	0.9045	0.0672	0.0572	0.0224	0.0262	0.9047	0.0682	0.0563	7.9695
	SE	0.2226	0.2865	0.7740	0.2871	0.2983	0.4872	0.4015	1.3715	0.4177	0.4554	0.4087
	CI size	0.0753	0.0611	0.2013	0.1516	0.1141	0.0845	0.0634	0.2057	0.1496	0.1141	1.7545
	CP	0.9790	0.9760	0.9610	0.9730	0.9680	0.9730	0.9670	0.9720	0.9790	0.9440	0.9239
	Bias	-0.0122	-0.0020	0.0255	-0.0022	0.0018	-0.0128	-0.0022	0.0253	-0.0032	0.0027	0.0305
	MSE	0.0001	0.0000	0.0007	0.0000	0.0000	0.0002	0.0000	0.0006	0.0000	0.0000	0.0009
$n = 1000$	Mean	0.0134	0.0251	0.9225	0.0651	0.0564	0.0137	0.0252	0.9211	0.0661	0.0577	7.9674
	SE	0.0160	0.0537	0.0640	0.0981	0.0590	0.0198	0.0521	0.0759	0.0901	0.0618	0.2888
	CI size	0.0299	0.0466	0.0818	0.1209	0.0902	0.0325	0.0476	0.0988	0.1188	0.0875	1.2196
	CP	0.9560	0.9590	0.9410	0.9570	0.9420	0.9510	0.9561	0.9440	0.9520	0.9492	0.9499
	Bias	-0.0037	-0.0011	0.0075	-0.0001	0.0026	-0.0041	-0.0012	0.0089	-0.0011	0.0013	0.0326
	MSE	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0011

4.2 Real data

In principle, price data are not available, since the call-on-max option is typically traded over-the-counter. For this reason, we cannot test the valuation models empirically. However, comparing models with different assumptions can be implemented, as in Zhang and Guegan (2008), Liu, Li and Ng (2015) and Chiou and Tsay (2008). In this section, we carry on the illustration of the proposed methodology on a real data set concerning the two stock prices of Brazilian companies. With the objective of analyzing two companies that could have a high correlation, we choose the companies Bradespar (BRAP4) and Vale S.A. (VALE3) with the aim of investigating two companies that could have a high correlation. The Brazilian company Bradespar admits the shareholdings that the bank Bradesco had in non-financial companies, among them: VCB, Vale, Scopus, and Globo. Thus, Bradespar's stocks price would be directly related to the stocks of Vale S.A., where the company holds the latter's stock control at 17.4 %. The analyzed period is from 07/01/2015 to 07/17/2018, containing 753 observations.

Figure 1 shows the high positive association between the two series, evidencing the requirement subject is financial options using these stocks, given its high correlation. Table 5 reports the similarity between the returns series, both concerning the minimum, mean, median, maximum, standard deviation (SD) and kurtosis, but the VALE3 series has a slightly more pronounced positive asymmetry than the BRAP4 series. As evidenced in section 2, asymmetry is present in financial series, a feature that symmetric GARCH processes have no potential to discriminate between positive and negative asymmetry.

Table 5. Descriptive statistics of returns.

Serie	Minimum	Mean	Median	Maximum	SD	Kurtosis	Skewness
BRAP4	-0.134	0.000	0.000	0.153	0.027	0.050	5.150
VALE3	-0.156	0.000	0.000	0.137	0.026	0.047	5.702

Before presenting the estimated coefficients of time series models, we focus on the analysis of the best model according to the selection criteria. Given the flexibility of the use of models based on copula functions, we select for each marginal the best model according to the selection criteria defined in Section 3.2. According to Table 6, all criteria corroborate that the model GARCH best fit the BRAP4 series, evidencing that there is no asymmetry present in this series, while, the best model for the VALE3 series is the EGARCH (evidencing the asymmetry). This result is in agreement with the statement in Table 5, where the VALE3 stock had an asymmetric coefficient more pronounced than BRAP4.

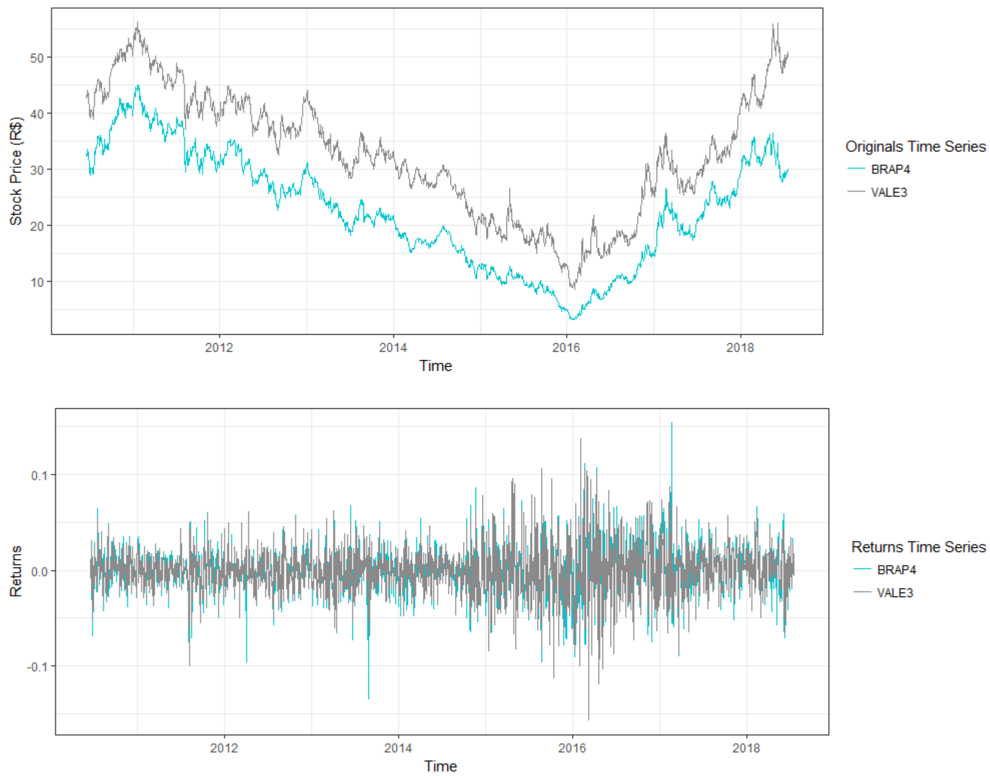

Figure 1. Original Series and Returns.

Table 6. Selection criteria for marginals.

BRAP4	GARCH	NGARCH	EGARCH	GJR-GARCH
AIC	-3071.3128	-3069.3172	-3070.1057	-3069.3678
AICc	-3071.2591	-3069.2367	-3070.0252	-3069.2872
CAIC	-3048.8271	-3041.2102	-3041.9987	-3041.2607
BIC	-3052.8271	-3046.2102	-3046.9987	-3046.2607
HQIC	-3064.1903	-3060.4142	-3061.2027	-3060.4647
VALE3	GARCH	NGARCH	EGARCH	GJR-GARCH
AIC	-3151.2693	-3150.0533	-3153.7289	-3151.7989
AICc	-3151.2156	-3149.9728	-3153.6484	-3151.7183
CAIC	-3123.6918	-3121.9463	-3128.7836	-3125.6219
BIC	-3128.6918	-3126.9463	-3132.7836	-3130.6219
HQIC	-3144.1468	-3141.1503	-3144.8258	-3142.8958

Table 7 reports the coefficients estimated via QML estimates and their respective robust standard errors. According to this result, we noticed that the best model for the BRAP4 series was the GARCH model, where it does not have an asymmetry parameter. We view in this model the high persistence, that is, $\alpha_1 + \beta_1$ very close to one, suggesting that the volatility can be persistent (strong temporal dependence), which opens options of models to analyze series with this feature. The best model for the VALE3 series was the EGARCH, where it presented a parameter of positive asymmetry, that is, a positive shock decreases its volatility.

Table 7. Estimated coefficients and corresponding robust standard errors for marginals.

BRAP4	GARCH	NGARCH	EGARCH	GJR-GARCH
$\hat{\alpha}_0$	6.9191e-06	6.8024e-06	-0.1399	7.0316e-06
	(4.8306e-06)	(4.7823e-06)	(0.0455)	(4.9614e-06)
$\hat{\alpha}_1$	0.0479	0.0477	0.1005	0.0507
	(0.0125)	(0.0127)	(0.0248)	(0.0175)
$\hat{\beta}$	0.9454	0.9457	0.9914	0.9446
	(0.0138)	(0.0140)	(0.0050)	(0.0144)
$\hat{\lambda}$	0.0568	0.0560	0.0560	0.0576
	(0.0358)	(0.0365)	(0.0359)	(0.0364)
$\hat{\gamma}$	-	0.0121	-0.0618	4.2924e-03
		(0.1628)	(0.0235)	(0.0180)
VALE3	**GARCH**	**NGARCH**	**EGARCH**	**GJR-GARCH**
$\hat{\alpha}_0$	3.7157e-06	3.0711e-06	-0.1081	2.5848e-06
	(2.9974e-06)	(2.9524e-06)	(0.0386)	(2.9020e-06)
$\hat{\alpha}_1$	0.0434	0.0428	0.0969	0.0555
	(0.0116)	(0.0111)	(0.0221)	(0.0152)
$\hat{\beta}$	0.9519	0.9522	0.9957	0.9554
	(0.0121)	(0.0117)	(0.0004)	(0.0113)
$\hat{\lambda}$	0.0579	0.0671	0.0762	0.0679
	(0.0357)	(0.0365)	(0.0387)	(0.0363)
$\hat{\gamma}$	-	0.1771	0.1433	0.0278
		(0.1854)	(0.1438)	(0.0171)

We consider the Kolmogorov-Smirnov, Jarque-Bera, Shapiro-Wilk, and Anderson-Darling tests to verify the assumption of normality of the residuals for the fitted models. Table 8 reports their p-values. All tests did not reject the null hypothesis at 5% that residuals follow a standard normal distribution. In addition, to verify that the increments are independent, Table 8 also reports the result of the Ljung-Box test, where, for all fitted models we do not reject the null hypothesis at 5 % that the residuals are independent.

Table 8. Tests of Normality and Independent Increments for residuals.

BRAP4	GARCH	NGARCH	EGARCH	GJR-GARCH
Kolmogorov-Smirnov	0.9315	0.9403	0.9514	0.9343
Jarque-Bera	0.1159	0.1142	0.2351	0.1225
Shapiro-Wilk	0.2571	0.2572	0.3802	0.2633
Anderson-Darling	0.6680	0.6725	0.6572	0.6652
Ljung-Box	0.4940	0.4938	0.4988	0.4944
VALE3	**GARCH**	**NGARCH**	**EGARCH**	**GJR-GARCH**
Kolmogorov-Smirnov	0.8737	0.8752	0.8761	0.8733
Jarque-Bera	0.2059	0.1680	0.2548	0.1433
Shapiro-Wilk	0.1752	0.1895	0.2644	0.1718
Anderson-Darling	0.2288	0.2627	0.3426	0.2697
Ljung-Box	0.1927	0.2079	0.2145	0.2152

Figure 2 shows the QQ-plots for the two best models for the series, that is, on the left panel is the GARCH for the BRAP4 series and on the right panel the EGARCH for the VALE3 series, corroborating with the tests in the Table 8, evidencing the non-rejection of the normality of the residuals.

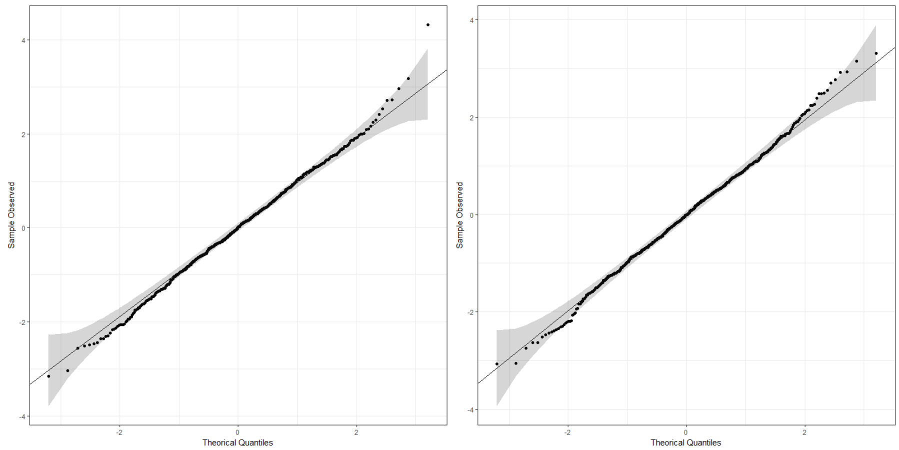

Figure 2. QQ-plots of residuals - GARCH BRAP4 (left panel) and EGARCH VALE3 (right panel).

Figure 3 illustrates the individual behavior of each set of residual fitted through the histograms and the joint behavior through the scatterplot in the center of the figure. As expected, the series has a highly positive association behavior, which is evidenced in the adjustment of the copulas given in Table 9, where the normal and Student-t copulas obtained high and positive values of their parameters (-1 $\leq \theta \leq$ 1).

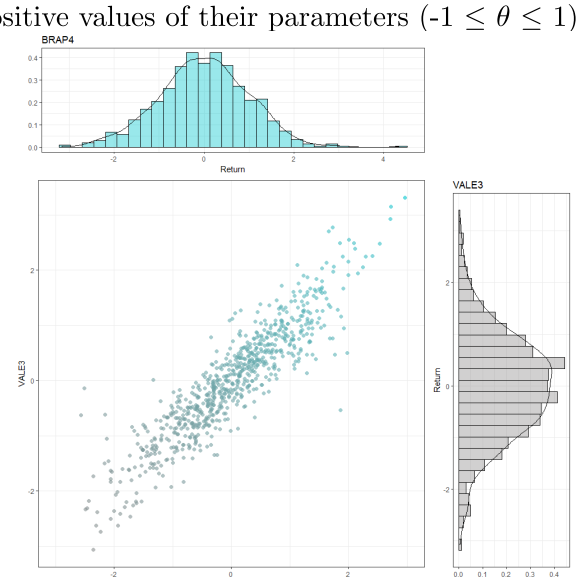

Figure 3. Scatterplot and histograms of residuals - GARCH BRAP4 and EGARCH VALE3.

Table 9. Estimated coefficients and corresponding standard errors (in parentheses) for copulas.

	Normal	Student-t	Gumbel	Frank	Joe
$\hat{\theta}$	0.9059	0.9133	3.4082	14.0430	4.0173
	(0.0048)	(0.0053)	(0.1040)	(0.4965)	(0.1423)

The degree of freedom of the Student-t copula and its respective standard deviation were 7.63401 and 1.7263.

According to Table 10 and the selection criteria adopted, the best copula for this data set was the Student-t copula, though the results found for the Student-t copula are very similar to the one observed for the Frank copula. The empirical copula and the copula adjusted by the diagonal method, where the excellent fit of the two copulas is noted, corroborate this result. The result of the Cramer Von Mises test are 0.0025, 0.0023, 0.0042, 0.0018 and 0.01122, for normal, Student-t, Gumbel, Frank and Joe copulas, respectively. As noted in

Figure 4, the result shows that Frank copula yields the smallest distance between fitted and empirical copula. We note that there is a minimal difference between the Frank and Student-t copulas. Therefore, these two copulas are considered in this work as the best fittings.

Table 10. Selection model of copulas.

	Normal	Student-t	Gumbel	Frank	Joe
AIC	-1290.6171	-1334.1487	-1231.5615	-1310.8730	-970.63696
AICc	-1290.6117	-1334.1327	-1231.5562	-1310.8676	-970.63162
CAIC	-1285.9957	-1324.9059	-1226.9401	-1306.2516	-966.01556
BIC	-1288.8365	-1330.5875	-1229.7809	-1309.0924	-968.85635
HQIC	-1284.9957	-1322.9059	-1225.9401	-1305.2516	-965.01556

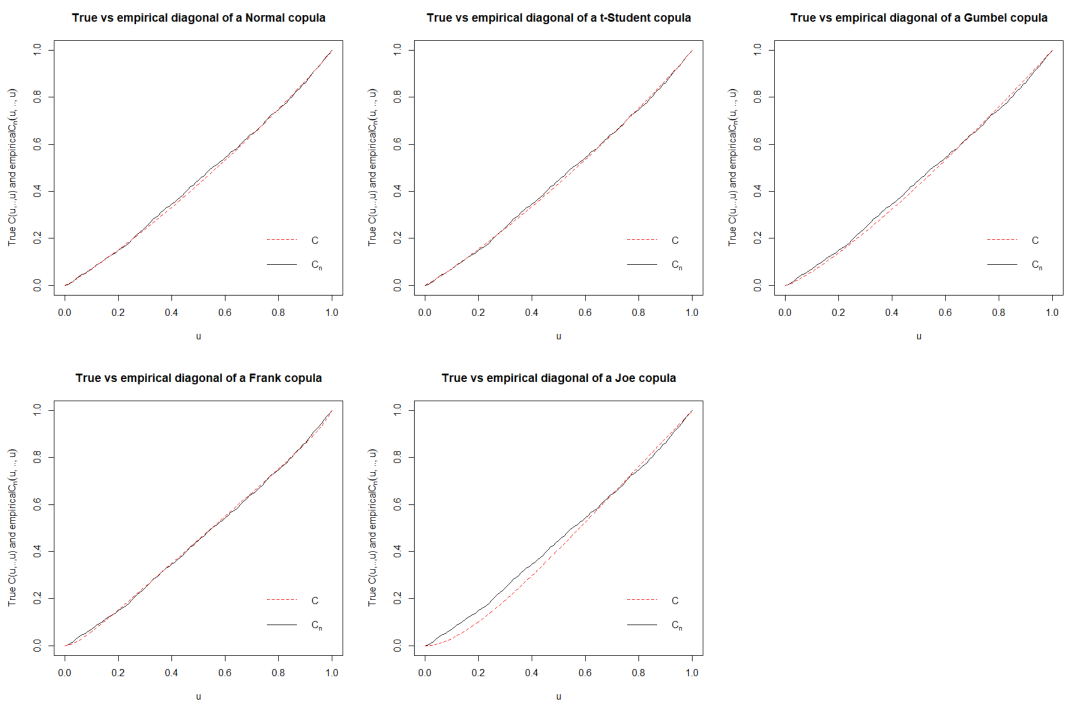

Figure 4. Comparing the empirical copula and the true copula on the diagonal.

Given the good fitting of the marginals obtained via time series models and the good joint fitting via copulas, we now calculate and analyze the option prices considering the call-on-max payoff function. To perform the comparison process, as a benchmark, we compare the results through the methodology proposed with the classical method, which is a Black-Scholes extension for the bivariate case (Haug, 2007), where this model considers the volatility constant over time and the linear dependence structure from the bivariate normal distribution.

The entire study was performed with 100 000 Monte Carlo simulations, 7 % interest rate and maturity time of one year. According to Table 9, as expected, the same behavior is observed for all models, that is, as the strike variable increases it is likely that, in a call option, the price of the option becomes cheaper. We note that the classical model obtained the lowest values for all strike values. Gesk and Roll (1984), Black (1975) and MacBeth and Merville (1980) corroborate this result for the univariate case, where the authors showed that the models that consider constant volatility over time underpricing the options, especially in-the-money (ITM) options. That is, a call option's strike price is below the market price in the univariate case. In this work we define ITM options when the strike price is less than the minimum between the two assets. Moreover, in Table 11, we can see that the Student-t and Frank copulas have the closest results to each other. The similarity in the excellent fit of the data can explain this result. We noticed the values obtained through normal copula obtained high results. The inability of the normal copula to capture observations in the tails of the distribution, a recurring fact in finances, can explain this result. The copula Joe obtained higher values mainly when the strike was smaller than 40, approaching the model of the normal copula. The Gumbel copula was the one that received the lowest values between the models. Figure 5 shows the behavior of the option price (z-axis) varying the maturity from 1 to 12 months (y-axis, in days) and strike (R$ 40.00 to R$ 60.00). We note that the higher the maturity the values differ little between strike prices, which does not happen when the option has a short maturity, where we indicate that setting at 50 maturity days there is a relatively significant difference varying the price of the strike. For example, Table 12 presents the prices for considering maturity = one month, six months and one year and strike = 20, 40 and 60.

Another fundamental aspect in the management of options risks is to know the levels of dependence between stocks. Therefore, Figure 6 presents the price behavior of the call-on-max option for the Student-t copula by varying its degrees of dependence. This result corroborates with those found by Chiou and Tsay (2008) for the call-on-max option using the American and Taiwanese indices. An intuitive interpretation is: the values of this option tend to be smaller when the underlying assets move in the same direction as when in opposite directions.

Table 11. Prices of a call-on-max option under various strikes values (R$).

Strike	Classic	Normal	Student-t	Gumbel	Frank	Joe
20	31.1182	32.3619	32.2648	32.2532	32.2711	32.5083
22	29.2693	30.5241	30.4283	30.4148	30.4329	30.6440
24	27.4764	28.7327	28.6402	28.6228	28.6425	28.8293
26	25.7468	26.9951	26.9054	26.8845	26.9045	27.0694
28	24.0867	25.3171	25.2295	25.2061	25.2258	25.3714
30	22.5003	23.7025	23.6169	23.5918	23.6110	23.7401
32	20.9906	22.1572	22.0733	22.0457	22.0646	22.1787
34	19.5594	20.6831	20.6028	20.5704	20.5889	20.6899
36	18.2071	19.2840	19.2055	19.1678	19.1867	19.2758
38	16.9331	17.9601	17.8810	17.8399	17.8602	17.9371
40	15.7360	16.7112	16.6316	16.5866	16.6093	16.6745
42	14.6140	15.5377	15.4571	15.4103	15.4349	15.4901
44	13.5643	14.4379	14.3578	14.3081	14.3346	14.3826
46	12.5842	13.4087	13.3304	13.2795	13.3062	13.3498
48	11.6705	12.4495	12.3723	12.3199	12.3477	12.3877
50	10.8198	11.5571	11.4799	11.4270	11.4576	11.4945
52	10.0288	10.7290	10.6522	10.5987	10.6321	10.6678
54	9.2941	9.9621	9.8872	9.8339	9.8675	9.9020
56	8.6122	9.2533	9.1807	9.1278	9.1613	9.1933
58	7.9798	8.6001	8.5283	8.4762	8.5101	8.5392
60	7.3937	7.9981	7.9271	7.8762	7.9117	7.9372

Table 12. Prices (R$) of a call-on-max option varying some Maturity time and Strike (R$).

Maturity\Strike	R$ 20.00	R$ 40.00	R$ 60.00
One Month	30.9671	10.9067	0.5537
Six Months	30.9190	13.6608	4.1864
One Year	30.7311	15.6953	7.0992

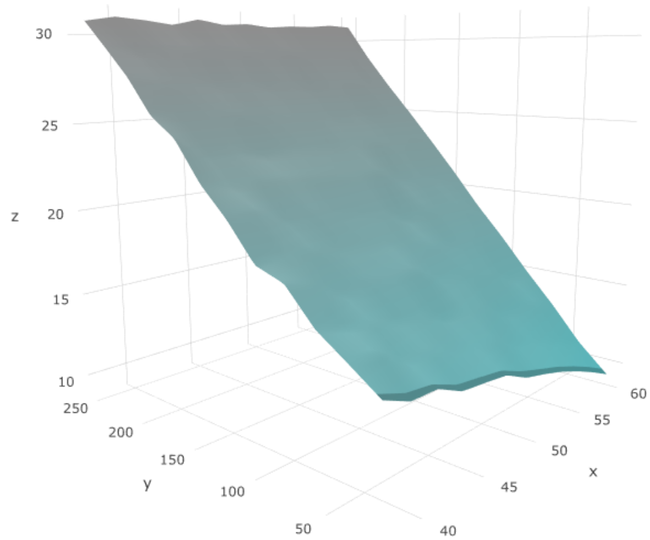

Figure 5. Price (R$) behavior of the call-on-max option ranging from Maturity to Strike.

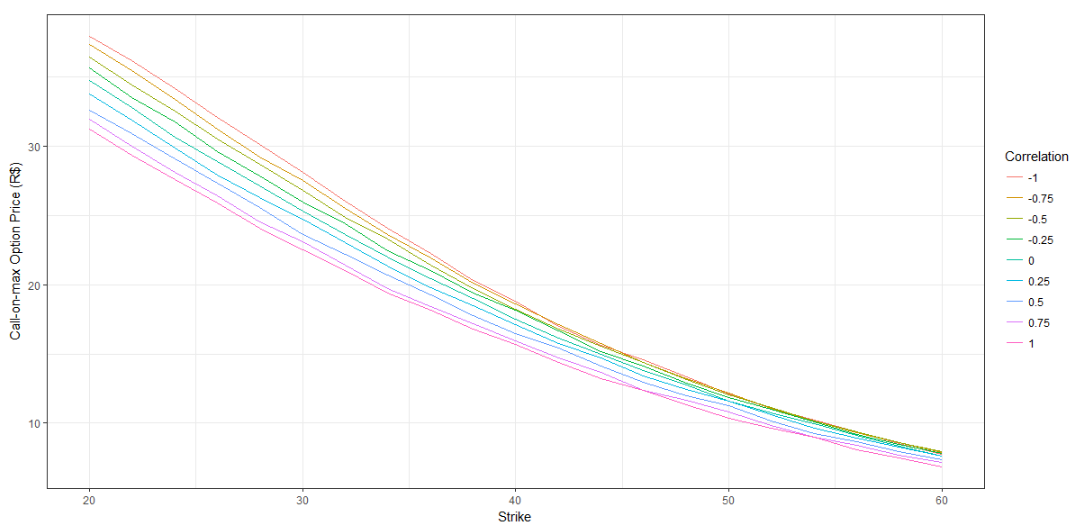

Figure 6. Behavior of the call-on-max option price by varying the copula parameter.

In addition, Figure 6 further shows that in-the-money options have the most substantial differences between dependency levels than out-the-money options (that is, when the strike is higher than the maximum between the two assets). Therefore, it was empirically verified the importance of a good joint fit of the stocks, and above all, the calculation of the correlation between the assets. Moreover, by employing the copulas functions, it is possible to capture linear, non-linear and caudal associations. Recalling, the traditional models derived from a Brownian geometric movement consider bivariate normal to price call-on-max options for two assets, and consequently, the linear correlation coefficient as the measure of association.

5. Concluding Remarks

In this paper, we propose an analysis and comparison among pricing models that consider the volatility of underlying assets and in the presence of dependence between copula framework. The model is an adequate methodology to realize a more realistic pricing option. To consider the modeling of asymmetry present in financial series, we examined three models that are extensions of the GARCH model under the neutral risk measure $\mathbb{Q}$, a pre-requisite to price options (NGARCH, EGARCH, and GJR-GARCH). Therefore, through the flexibility of the copula functions, we chose which marginal processes fit best with each stock and thus proceeded in the joint fitted.

Two databases illustrate the application of the methodology. The first one was an artificial database with the objective of carrying out a simulation study and the second a database of two Brazilian companies. The simulation study showed that all models presented good asymptotic properties. In addition, in the real time-series of two Brazilian stock companies, the model offered a proper fitting and the results obtained were confronted with the classic model, which is an extension of the Black-Scholes model.

The contributions of the proposed method in the present paper are as follows: (i) using the best copula makes the model more suitable; (ii) extension to marginal models that consider asymmetry makes joint modeling more flexible and realistic; (iii) a comparison of methodologies highlights the role of risk management; (iv) due to the good marginal and joint fitted, in addition to the values obtained in relation to the classical consolidated model, there are arguments to believe that the differences obtained between the best models, through the copulas and the extension of the conventional method, are improvements in the calculation of the fair value; and (v) the empirical relevance of such alternatives is apparent given the evidence of non-joint-normality in financial emerging markets.

Finally, we highlight some points for future work. The first one of them, even with extensions to asymmetric models, we often have financial series with heavy tails, which should derive a risk-neutral measure Q for these models, such as considering the non-normality of the residuals. The second point is the adoption of other copula functions, such as power variance function family copulas. The third, we can consider another skew distribution for the errors, such as Arrellano-Valle et al. (2010), Minozzo et al. (2012) and Marcos et al. (2012).

Appendix

The normal copula

The normal copula or commonly known as Gaussian copula receives this name because it comes from the normal density function for $d \geq 2$. A normal bivariate copula is expressed by

$$C(u,v) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{1}{2\pi\sqrt{1-\rho^2}} \mathrm{e}^{-\dfrac{t_1^2 - 2\rho t_1 t_2 + t_2^2}{2(1-\rho^2)}} \mathrm{d}t_1^2 \mathrm{d}t_2^2,$$

where $x_1 = \Phi^{-1}(u)$, $x_2 = \Phi^{-1}(v)$, for $-1 \leq \rho \leq 1$. This type of copula has no dependence on the tails of the distributions and is symmetric.

The Student-t copula

The Student-t copula coincides with the bivariate Student-t distribution function, where its form is defined as

$$C(u,v) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{t_1^2-2\rho t_1 t_2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2} \mathrm{d}t_1\mathrm{d}t_2,$$

where ν represents the degrees of freedom of Student-t distribution. As in the case of normal copula, the Student-t marginal copula coincides with the Student-t standard, being $x_1 = t_\nu^{-1}(u)$ e $x_2 = t_\nu^{-1}(v)$. This type of copula does not have independence in the tails, which favors its use in extreme events, such as, for example, unplanned oscillations in the stock market. However, given the symmetry of the function, the degree of dependence on the upper tail is equal to the lower tail.

The Gumbel copula

The Gumbel copula is characterized by the dependence only on the upper tail and is represented as

$$C(u,v) = \mathrm{e}^{-[(-\log(u))^\theta+(-\log(v))^\theta]^{1/\theta}},$$

where $\theta \in [1,\infty]$. When $\theta \to \infty$, dependence is perfectly positive and independent when $\theta = 1$.

The Frank Copula

The form of a Frank copula is expressed through

$$C(u,v) = -\frac{1}{\theta}\log\left(1+\frac{[\mathrm{e}^{-\theta u}-1][\mathrm{e}^{-\theta v}-1]}{\mathrm{e}^{-\theta}-1}\right)$$

where $\theta \neq 0$. When $\theta \to \infty$, we have perfect positive dependence and we have the case of independence when we $\theta \to 0$. This copula has the same dependence on both function tails, such as elliptic copulas.

The Joe copula

The Joe copula is given by

$$C(u,v) = 1-\Big([1-u]^\theta+[1-v]^\theta-[1-u]^\theta[1-v]^\theta\Big)^{1/\theta},$$

where $1 \leq \theta \leq \infty$. When $\theta = 1$, we have the case of independence and the case of perfect positive dependence when $\theta \to \infty$.

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