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Research Paper

A new extension of power Lindley distribution for analyzing bimodal data

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Abstract

In this article, we introduce a new three-parameter odd log-logistic power Lindley distribution and discuss some of its properties. These include the shapes of the density and hazard rate functions, mixture representation, the moments, the quantile function, and order statistics. Maximum likelihood estimation of the parameters and their estimated asymptotic standard errors are derived. Three algorithms are proposed for generating random data from the proposed distribution. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. An application of the model to two real data sets is presented finally and compared with the fit attained by some other well-known two and three-parameter distributions for illustrative purposes. It is observed that the proposed model has some advantages in analyzing lifetime data as compared to other popular models in the sense that it exhibits varying shapes and shows more flexibility than many currently available distributions.

1. Introduction

The Lindley distribution specified by the probability density function (pdf)

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x), \quad x > 0, \quad \theta > 0.$$

was introduced by Lindley (1958) in the context of Bayesian statistics. Ghitany et al. (2008) investigated properties of the Lindley distribution with application and outlined that the Lindley distribution is a better model than one based on the exponential distribution, in other words many mathematical properties of the Lindley distribution are more flexible than those of the exponential distribution. Ghitany et al. (2008) showed that the Lindley distribution can be written as a mixture of an exponential distribution and a gamma distribution with shape parameter 2.

Gleaton and Lynch (2004, 2006) introduced a new family of distributions which is called "the Generalized log-logistic family of distributions". The cumulative distribution function

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(cdf) of this family is given by

$$F(x;\alpha,\boldsymbol{\xi}) = \frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha} + \overline{G}(x;\boldsymbol{\xi})^{\alpha}}, \tag{1}$$

where $\alpha > 0$ is the shape parameter, $G(x;\boldsymbol{\xi})$ is the cdf of the baseline distribution, $\overline{G}(x;\boldsymbol{\xi}) = 1 - G(x;\boldsymbol{\xi})$ is the survival function and $\boldsymbol{\xi}$ is the set of parameters of the baseline distribution $G(\cdot)$. In addition, the pdf of the family is

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{\alpha\, g(x;\boldsymbol{\xi})\, G(x;\boldsymbol{\xi})^{\alpha-1}\, \overline{G}(x;\boldsymbol{\xi})^{\alpha-1}}{\left[G(x;\boldsymbol{\xi})^{\alpha} + \overline{G}(x;\boldsymbol{\xi})^{\alpha}\right]^2}.$$

This family was called later the odd log-logistic family of distributions. If the baseline distribution possesses a closed form cdf, the generated new distribution will also possess a closed form cdf. One can easily show that

$$\frac{\log\left[\frac{F(x;\alpha,\boldsymbol{\xi})}{\overline{F}(x;\alpha,\boldsymbol{\xi})}\right]}{\log\left[\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right]} = \alpha.$$

Therefore α is the quotient of the log-odds ratio for the generated and baseline distributions.

Recently, Ghitany et al. (2013) proposed a generalization of the Lindley distribution, the power Lindley distribution, with cdf

$$G(x) = 1 - \left(1 + \frac{\lambda}{1+\lambda}\, x^{\beta}\right)\mathrm{e}^{-\lambda x^{\beta}}, \quad x > 0, \beta, \lambda > 0.$$

Now, by letting $G(x;\boldsymbol{\xi})$ in (1) to be the cdf of the power Lindley distribution, where $\boldsymbol{\xi} = (\beta,\lambda)$ is the set of parameters, we can obtain a new extension of the power Lindley distribution, called the odd log-logistic power Lindley (henceforth, OLL-PL) distribution. The cdf, pdf and hazard rate function of this distribution are given by

$$F(x;\alpha,\beta,\lambda) = \frac{\left[1 - (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha}}{\left[1 - (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha} + (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})^{\alpha}\, \mathrm{e}^{-\lambda\alpha\, x^{\beta}}}, \tag{2}$$

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\, \lambda^2\, x^{\beta-1}\, (1 + x^{\beta})\, \mathrm{e}^{-\alpha\,\lambda x^{\beta}} \left[1 - (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha-1} (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})^{\alpha-1}}{(1+\lambda)\left\{\left[1 - (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha} + (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})^{\alpha}\, \mathrm{e}^{-\lambda\alpha\, x^{\beta}}\right\}^2},$$
$$x > 0, \tag{3}$$

and

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\, \lambda^2\, x^{\beta-1}\, (1 + x^{\beta}) \left[1 - (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha-1}}{(1+\lambda)\, (1 + \frac{\lambda}{1+\lambda}\, x^{\beta}) \left\{\left[1 - (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha} + (1 + \frac{\lambda}{1+\lambda}\, x^{\beta})^{\alpha}\, \mathrm{e}^{-\lambda\alpha\, x^{\beta}}\right\}}, \tag{4}$$

respectively, where $\alpha, \beta, \lambda > 0$. We write $X \sim$ OLL-PL(α, β, λ) if the pdf of X can be written as (3). The new distribution is very flexible in the sense that it can be skewed and symmetric depending upon the specific choices of the parameters. Furthermore, the associated cdf is in closed form. Consequently, this distribution can be applied to modelling censored data too. This is a major motivation to carry out this work. Furthermore, in reliability engineering and lifetime analysis, we often assume that the failure times of the components within each system follow the exponential lifetimes; see, for example, Adamidis and Loukas (1998) among others and the references therein. This assumption may seem unreasonable because, for the exponential distribution, the hazard rate is a constant, whereas many real-life systems do not have constant hazard rates, and the components of a system are often more rigid than the system itself, such as bones in a human body, balls of a steel pipe, etc. Accordingly, it becomes reasonable to consider the components of a system to follow a distribution with a non-constant hazard function that has flexible hazard function shapes.

From Figure 1, we see that this model can be bimodal ($\alpha = 0.5, \beta = 3, \lambda = 2$) which is rare in classic lifetime distributions. We can also observe that when α and λ are fixed, then for small values of β, the density is often decreasing. However, for larger values of β, the density may not be decreasing any longer and it can be unimodal when $\alpha = 1.5$ and 3 and unimodal, bimodal or decreasing-increasing-decreasing when $\alpha = 0.1$ and 0.5. In addition, it seems that λ affects the height of the density plots.

From Figure 2, we can see that the hazard function is often decreasing for small values of β. For larger values of β, the hazard function can be increasing (for example for the case ($\alpha = 0.5, \beta = 3, \lambda = 2$)) and bath-tub shaped (for example for the case ($\alpha = 0.1, \beta = 3, \lambda = 2$)). In addition, the hazard function can be upside-down bath-tub shaped too (for example for the case ($\alpha = 3, \beta = 0.5, \lambda = 2$)).

In addition, as we will see in subsection 2.4, the additional parameter α plays the role of controlling the tail weights of the new distribution.

An interpretation of the OLL-PL distribution can be given as follows: Let X be a lifetime random variable having power Lindley distribution. The odds ratio that an individual (or component) following the lifetime X will die (fail) at time x is $y = G(x; \beta, \lambda)/\bar{G}(x; \beta, \lambda)$. Here, one can consider this odds of death as a random variable, say Y. Now, if we model the randomness of the "odds of death" using the log-logistic distribution with scale parameter 1 and shape parameter α, ($F_Y(y) = y^\alpha/[1 + y^\alpha]$ for $y > 0$), then we can write

$$Pr(Y \leq y) = F_Y\left(G(x; \beta, \lambda)/\bar{G}(x; \beta, \lambda)\right),$$

which is given by (2), see Cooray (2006) for more details regarding this interpretation.

Special Cases:

- For $\alpha = 1$, we obtain the power Lindley distribution.
- For $\beta = 1$, we obtain the odd log-logistic Lindley distribution (Ozel et al., 2016).
- For $\alpha = \beta = 1$, we obtain the Lindley distribution.

We hope that this new distribution can be applied to describing lifetime data more properly than the existing distributions. The major motivation of introducing the OLL-PL distribution can be summarized as follows. (i) The OLL-PL distribution contains several lifetime distributions as special cases, such as the power Lindley (PL) distribution due to Ghitany et al. (2013) for $\alpha = 1$. (ii) It is shown in Section 2 that the OLL-PL distribution can be viewed as a mixture of exponentiated power Lindley (EPL) distributions introduced by Warahena-Liyanage and Pararai (2014) and Ashour and Eltehiwy (2015). (iii) The OLL-PL distribution is a flexible model which can be widely used for modeling lifetime data. (iv) The OLL-PL distribution exhibits monotone as well as non-monotone hazard rates

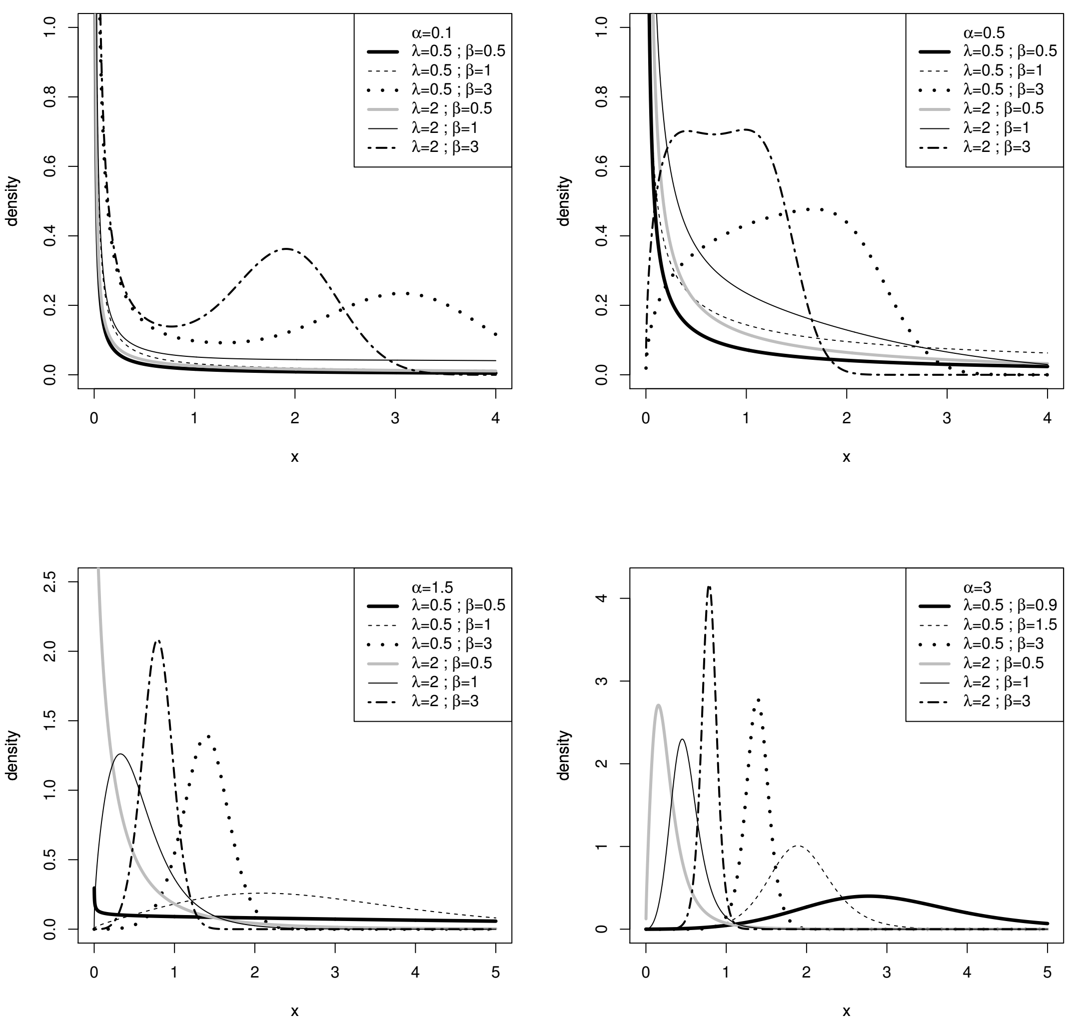

Figure 1. Pdfs of the OLL-PL model for selected α, β and λ.

but does not exhibit a constant hazard rate, which makes this distribution to be superior to other lifetime distributions, which exhibit only monotonically increasing/decreasing, or constant hazard rates. (v) The OLL-PL distribution outperforms several of the well-known lifetime distributions with respect to some real data examples.

The rest of the article is organized as follows: In Section 2, we discuss some structural properties of the OLL-PL distribution. Section 3 deals with the classical method of estimation (using maximum likelihood) of the model parameters of the OLL-PL distribution, and a small simulation study is conducted to verify the efficacy of the said estimation procedure. In Section 4, two real data sets are considered as an example to illustrate the applicability of OLL-PL distribution. In Section 5, we provide some concluding remarks.

2. Structural properties

In this section, we discuss some structural properties of the OLL-PL distribution.

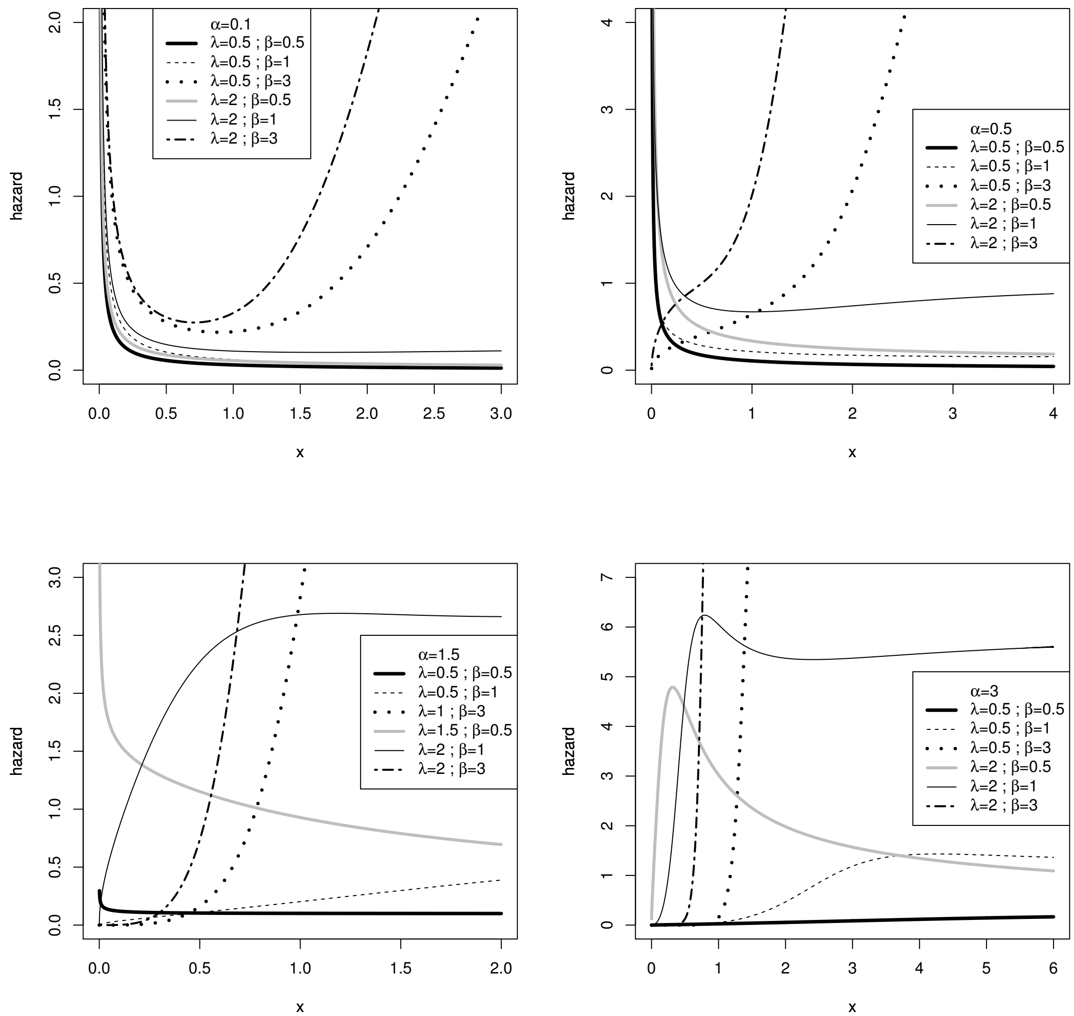

Figure 2. Hazard rate functions of the OLL-PL model for selected α, β and λ.

2.1 Mixture representations for the pdf and cdf

The EPL distribution, introduced by Warahena-Liyanage and Pararai (2014) and Ashour and Eltehiwy (2015), has the pdf

$$f_{EPL}(x;\alpha,\beta,\lambda) = \frac{\alpha\lambda^2}{\lambda+1}(1+x^{\beta})x^{\beta-1}\mathrm{e}^{-\lambda x^{\beta}}\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha-1},\ x>0,\alpha,\beta,\lambda>0. \quad (5)$$

We write $X \sim \mathrm{EPL}(\alpha,\beta,\lambda)$ if the pdf of X can be expressed as (5). In addition, the cdf of the EPL model is

$$F_{EPL}(x;\alpha,\beta,\lambda) = \left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha}, \quad x>0.$$

Now, we show that the OLL-PL distribution can be viewed as a mixture of EPL distributions. Using the generalized binomial expansion, the numerator of (2) can be written as

$$\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha}=\sum_{k=0}^{\infty}a_k\,\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{k},$$

where $a_k=\sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}$ and the denominator of (2) can be written as

$$\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha}+(1+\frac{\lambda}{1+\lambda}\,x^{\beta})^{\alpha}\,\mathrm{e}^{-\lambda\alpha\,x^{\beta}}=\sum_{k=0}^{\infty}b_k\,\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{k},$$

where $b_k=a_k+(-1)^k\binom{\alpha}{k}$. Therefore the cdf of the OLL-PL distribution can be expressed as

$$F(x)=\frac{\sum_{k=0}^{\infty}a_k\,\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{k}}{\sum_{k=0}^{\infty}b_k\,\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{k}}=\sum_{k=0}^{\infty}c_k\,\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{k},$$

where $c_0=\frac{a_0}{b_0}=0$ and for $k\geq 1$ we have

$$c_k=b_0^{-1}\left[a_k-b_0^{-1}\sum_{r=1}^{k}b_r\,c_{k-r}\right].$$

Or equivalently, we can write the cdf of OLL-PL as

$$F(x)=\sum_{k=1}^{\infty}c_kF_{EPL}(x;k,\beta,\lambda)=\sum_{k=0}^{\infty}c_{k+1}F_{EPL}(x;k+1,\beta,\lambda), \tag{6}$$

where $F_{EPL}(x;k+1,\beta,\lambda)$ denotes the cdf of the EPL distribution with parameters $k+1$, β and λ. We note that $\sum_{k=0}^{\infty}c_{k+1}=1$.

By differentiating equation (6), the pdf of the OLL-PL distribution can be expanded as

$$f(x)=\sum_{k=0}^{\infty}c_{k+1}\,f_{EPL}(x;k+1,\beta,\lambda),$$

where $f_{EPL}(x;k+1,\beta,\lambda)$ denotes the pdf of the EPL distribution with parameters $k+1$, β and λ.

2.2 Moments

We define and compute

$$A(a_1,a_2,a_3;\beta,\lambda)=\int_0^{\infty}x^{a_1}\,(1+x^{\beta})\,\mathrm{e}^{-a_2\,x^{\beta}}\,\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{a_3}\,\mathrm{d}x,$$

where $a_1>-1, a_2>0$ and $a_3\in$ R.

Using the generalized binomial expansion, one can obtain

$$A(a_1, a_2, a_3; \beta, \lambda) = \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{l_1}\sum_{l_3=0}^{l_2+1}\binom{a_3}{l_1}\binom{l_1}{l_2}\binom{l_2+1}{l_3}\frac{(-1)^{l_1}\lambda^{l_2}\,\Gamma(\frac{a_1+1}{\beta}+l_3)}{\beta\,(\lambda+1)^{l_1}\,(\lambda\, l_1+a_2)^{\frac{a_1+1}{\beta}+l_3}}.$$

Next, the m-th moment of the OLL-PL distribution will be

$$E\,[X^m] = \frac{\beta\,\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,A(m+\beta-1, \lambda, k; \beta, \lambda).$$

For integer values of m, let $\mu'_m = E(X^m)$ and $\mu = \mu'_1 = E(X)$, then one can also find the m-th central moment of the OLL-PL distribution through the following well-known equation

$$\mu_m = E(X-\mu)^m = \sum_{r=0}^{m}\binom{m}{r}\mu'_r(-\mu)^{m-r}. \quad (7)$$

Using (7), the variance, skewness and kurtosis measures can be obtained from the following relations:

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - [E(X)]^2,\\
Skewness(X) &= \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{[Var(X)]^{\frac{3}{2}}},\\
Kurtosis(X) &= \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4}{[Var(X)]^2}.
\end{aligned}$$

Figure 3 shows the behavior of the skewness and kurtosis of the OLL-PL distribution with respect to α and λ when $\beta = 2$.

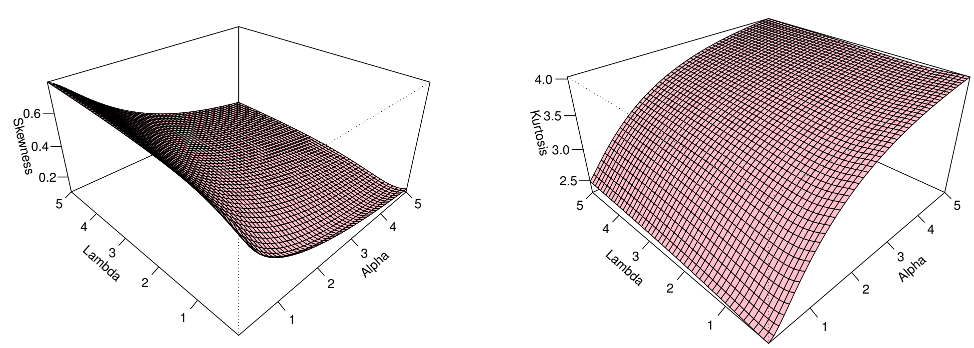

Figure 3. Skewness (left) and kurtosis (right) plots when $\beta = 2$.

Next, we define and compute

$$B(a_1, a_2, a_3, y; \beta, \lambda) = \int_0^y x^{a_1}\,(1+x^\beta)\,\mathrm{e}^{-a_2\,x^\beta}\left[1-(1+\frac{\lambda}{1+\lambda}\,x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^{a_3}\mathrm{d}x,$$

where $a_1 > -1, a_2 > 0$ and $a_3 \in \mathrm{R}$.

Using generalized binomial expansion, one can obtain

$$B(a_1, a_2, a_3, y; \beta, \lambda) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{l_1} \sum_{l_3=0}^{l_2+1} \binom{a_3}{l_1} \binom{l_1}{l_2} \binom{l_2+1}{l_3} \frac{(-1)^{l_1} \lambda^{l_2} \, \gamma(\frac{a_1+1}{\beta} + l_3, (\lambda\, l_1 + a_2)\, y^{\beta})}{\beta\, (\lambda+1)^{l_1}\, (\lambda\, l_1 + a_2)^{\frac{a_1+1}{\beta}+l_3}},$$

where $\gamma(\lambda, z) = \int_0^z t^{\lambda-1}\, \mathrm{e}^{-t}\, \mathrm{d}t$ denotes the incomplete gamma function.

Now, the r-th incomplete moment of the OLL-PL distribution is found to be

$$m_r(y) = \int_0^y x^r\, f(x) \mathrm{d}x = \frac{\lambda^2\, \beta}{1+\lambda} \sum_{k=0}^{\infty} (k+1)\, c_{k+1}\, B(r+\beta-1, \lambda, k, y; \beta, \lambda).$$

2.3 Stochastic ordering

Let X and Y be two random variables. X is said to be stochastically less than or equal to Y, denoted by $X \underset{st}{\leqslant} Y$ if $P(X > x) \leq P(Y > x)$ for all x in the support set of X.

Theorem 2.1 Suppose $X \sim$ OLL-PL$(\alpha_1, \beta_1, \lambda)$ and $Y \sim$ OLL-PL$(\alpha_2, \beta_2, \lambda)$. If $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, then $X \underset{st}{\leqslant} Y$.

Proof Straightforward and hence omitted.

2.4 Quantile Function

Quantile functions are in widespread use in statistics and often find representations in terms of lookup tables for key percentiles. Let $X \sim$ OLL-PL(α, β, λ). The quantile function of X, say $Q(p)$, is defined by $F(Q(p)) = p$ and is the root of the following equation

$$(1 + \lambda + \lambda\, Q(p)^{\beta})\, \mathrm{e}^{-\lambda\, Q(p)^{\beta}} = \frac{(1+\lambda)\,(1-p)^{\frac{1}{\alpha}}}{p^{\frac{1}{\alpha}} + (1-p)^{\frac{1}{\alpha}}}, \qquad (8)$$

for $0 < p < 1$. Inserting $Z(p) = -1 - \lambda - \lambda\, Q(p)^{\beta}$, into (8), we have

$$Z(p)\, \mathrm{e}^{Z(p)} = \frac{-(1+\lambda)\,(1-p)^{\frac{1}{\alpha}}\, \mathrm{e}^{-1-\lambda}}{p^{\frac{1}{\alpha}} + (1-p)^{\frac{1}{\alpha}}}.$$

Hence, the solution of $Z(p)$ is

$$Z(p) = W_{-1}\left[\frac{-(1+\lambda)\,(1-p)^{\frac{1}{\alpha}}\, \mathrm{e}^{-1-\lambda}}{p^{\frac{1}{\alpha}} + (1-p)^{\frac{1}{\alpha}}}\right],$$

where $W_{-1}[.]$ is the negative branch of the Lambert W function (Corless et al. 1996). Thus, we obtain the quantile function as

$$Q(p) = \left\{-1 - \frac{1}{\lambda} - \frac{1}{\lambda}\, W_{-1}\left[\frac{-(1+\lambda)\,(1-p)^{\frac{1}{\alpha}}\, \mathrm{e}^{-1-\lambda}}{p^{\frac{1}{\alpha}} + (1-p)^{\frac{1}{\alpha}}}\right]\right\}^{\frac{1}{\beta}}. \qquad (9)$$

The particular case of (9) for $\alpha = \beta = 1$ has been derived recently by Jodrá (2010).

Here, we also propose three different algorithms for generating random data from the OLL-PL distribution. The first algorithm is based on generating random data from the Lindley distribution using the exponential-gamma mixture (see Ghitany et al., 2008).

- **Algorithm 1 (Mixture Form of the Lindley Distribution)**

  (1) *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n;$
  (2) *Generate* $V_i \sim Exponential(\lambda), i = 1, \cdots, n;$
  (3) *Generate* $W_i \sim Gamma(2, \lambda), i = 1, \cdots, n;$
  (4) *If* $\frac{U_i^{\frac{1}{\alpha}}}{U_i^{\frac{1}{\alpha}} + (1-U_i)^{\frac{1}{\alpha}}} \leq \frac{\lambda}{1+\lambda}$ *set* $X_i = V_i^{\frac{1}{\beta}}$, *otherwise, set* $X_i = W_i^{\frac{1}{\beta}}, i = 1, \cdots, n.$

The second algorithm is based on generating random data using the Weibull-generalized gamma (GG) mixture (see Ghitany et al., 2013).

- **Algorithm 2 (Mixture Form of the power Lindley Distribution)**

  (1) *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n;$
  (2) *Generate* $Y_i \sim Weibull(\beta, \lambda), i = 1, \cdots, n;$
  (3) *Generate* $Z_i \sim GG(2, \beta, \lambda), i = 1, \cdots, n;$
  (4) *If* $\frac{U_i^{\frac{1}{\alpha}}}{U_i^{\frac{1}{\alpha}} + (1-U_i)^{\frac{1}{\alpha}}} \leq \frac{\lambda}{1+\lambda}$ *set* $X_i = Y_i$, *otherwise, set* $X_i = Z_i, i = 1, \cdots, n.$

The third algorithm is based on generating random data from the inverse cdf in (2) of the OLL-PL distribution, see (9).

- **Algorithm 3 (Inverse cdf)**

  (1) *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n;$
  (2) *Set*

$$X_i = \left\{ -1 - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1} \left[\frac{-(1+\lambda)\,(1-U_i)^{\frac{1}{\alpha}}\,\mathrm{e}^{-1-\lambda}}{U_i^{\frac{1}{\alpha}} + (1-U_i)^{\frac{1}{\alpha}}} \right] \right\}^{\frac{1}{\beta}}, i = 1, \cdots, n.$$

Algorithm 1 is the simplest data generation algorithm and therefore is preferable. Data generation from the classical distributions like uniform, exponential and gamma distributions is included normally in many statistical software. We have used Algorithm 1 in our simulation study, see Subsection 3.1. Algorithm 3 involves the Lambert W function and therefore is somehow complicated, see Corless et al. (1996) for more details regarding the Lambert W function.

2.5 Asymptotic properties

Let $X \sim$ OLL-PL(α, β, λ), then the asymptotics of equations (2), (3) and (4) as $x \to 0$ are given by

$$F(x) \sim (\lambda\, x^{\beta})^{\alpha} \quad \text{as } x \to 0,$$
$$f(x) \sim \alpha\, \beta\, \lambda^{\alpha}\, x^{\alpha\beta-1} \quad \text{as } x \to 0,$$
$$h(x) \sim \alpha\, \beta\, \lambda^{\alpha}\, x^{\alpha\beta-1} \quad \text{as } x \to 0.$$

The asymptotics of equations (2), (3) and (4) as $x \to \infty$ are given by

$$1-F(x)\sim(\frac{\lambda}{1+\lambda})^{\alpha}\,x^{\alpha\,\beta}\,\mathrm{e}^{-\alpha\lambda\,x^{\beta}}\quad\text{as }\ x\to\infty,$$
$$f(x)\sim\alpha\,\lambda\,\beta(\frac{\lambda}{1+\lambda})^{\alpha}\,x^{\beta(\alpha+1)-1}\,\mathrm{e}^{-\alpha\lambda\,x^{\beta}}\quad\text{as }\ x\to\infty,$$
$$h(x)\sim\alpha\beta\lambda\,x^{\beta-1}\quad\text{as }\ x\to\infty.$$

These equations show the effect of parameters on the tails of the OLL-PL distribution.

2.6 Extreme values

Let $X_1,\ldots,X_n$ be a random sample from (3) and $\bar{X}=(X_1+\cdots+X_n)/n$ denote the sample mean, then by the usual central limit theorem, the distribution of $\sqrt{n}(\bar{X}-E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n\to\infty$. Sometimes one would be interested in the asymptotics of the extreme values $M_n=\max(X_1,\ldots,X_n)$ and $m_n=\min(X_1,\ldots,X_n)$. For (2), it can be seen that

$$\lim_{t\to 0}\frac{F(t\,x)}{F(t)}=x^{\alpha\beta},$$

and

$$\lim_{t\to\infty}\frac{1-F(t\,x)}{1-F(t)}=\mathrm{e}^{-\alpha\lambda\,x^{\beta}}.$$

Thus, it follows from Theorem 1.6.2 in Leadbetter et al. (1983) that there must be normalizing constants $a_n>0, b_n, c_n>0$ and d_n such that

$$Pr\,[a_n(M_n-b_n)\le x]\to\mathrm{e}^{-\mathrm{e}^{-\lambda\alpha\,x^{\beta}}},$$

and

$$Pr\,[a_n(m_n-b_n)\le x]\to 1-\mathrm{e}^{-x^{\alpha\beta}},$$

as $n\to\infty$. Using Corollary 1.6.3 of Leadbetter et al. (1983), we can obtain the form of normalizing constants a_n, b_n, c_n and d_n.

2.7 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose that $X_1,\ldots,X_n$ are a random sample from an OLL-PL distribution. Let $X_{i:n}$ denote the i-th order statistic. The pdf of $X_{i:n}$ can be expressed as (see Arnold et al., 1992)

$$f_{i:n}(x)=\,K\,f(x)\,F^{i-1}(x)\,\,\{1-F(x)\}^{n-i}=K\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\,f(x)\,F(x)^{j+i-1},$$

where $K=\frac{n!}{(i-1)!\,(n-i)!}$.

We use the result 0.314 of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n (for $n \geq 1$)

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} d_{n,i}\, u^i,$$

where the coefficients $d_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, d_{n,i-m}.$$

We can demonstrate that the density function of the i-th order statistic of an OLL-PL distribution can be expressed as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} \sum_{j=0}^{n-i} m^{*}_{r,k,j}\, f_{EPL}(x; r+k+i+j, \beta, \lambda), \tag{10}$$

where $f_{EPL}(x; \alpha, \beta, \lambda)$ denotes the density function of EPL distribution with parameters α, β and λ and the coefficients $m^{*}_{r,k,j} \equiv m^{*}_{r,k,j}(i,n)$'s are given by

$$m^{*}_{r,k,j} = \frac{n!\,(r+1)\, c_{r+1}\, (-1)^j\, a^{*}_{j+i-1,k}}{(i-1)!\,(n-i-j)!\, j!\,(r+k+i+j)},$$

in which the coefficients c_r's are defined in Subsection 2.1 and the quantities $a^{*}_{j+i-1,k}$ can be determined such that $a^{*}_{j+i-1,0} = c_1^{j+i-1}$ and for $k \geq 1$

$$a^{*}_{j+i-1,k} = (k\, c_1)^{-1} \sum_{q=1}^{k} [q\,(j+i) - k]\, c_{q+1}\, a^{*}_{j+i-1,k-q}.$$

Equation (10) is the main result of this section. It reveals that the pdf of the OLL-PL order statistic is a linear combination of EPL distributions. So, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, moment generating function, mean deviations and others can be derived using this result.

3. Estimation

In this section, we discuss maximum likelihood estimation (MLE) and inference for the OLL-PL distribution. Let $x_1, \ldots, x_n$ be a random sample from OLL-PL model where α, β and λ are the unknown parameters. The log-likelihood for the parameters of the OLL-PL distribution given the data set $x_1, \ldots, x_n$ reduces to

$$l_n = n \log\left(\frac{\alpha\, \beta\, \lambda^2}{1+\lambda}\right) + (\beta - 1) \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \log(1 + x_i^{\beta}) - \lambda \sum_{i=1}^{n} x_i^{\beta}$$
$$+ (\alpha - 1) \sum_{i=1}^{n} \log\left[t_i(1-t_i)\right] - 2 \sum_{i=1}^{n} \log\left[t_i^{\alpha} + (1-t_i)^{\alpha}\right], \tag{11}$$

where $t_i = 1 - \left(1 + \frac{\lambda}{1+\lambda}\, x_i^{\beta}\right) \mathrm{e}^{-\lambda x_i^{\beta}}$.

The associated nonlinear log-likelihood system $\frac{\partial l_n}{\partial h} = 0$, (for $h = \alpha, \beta, \lambda$), follows as

$$
\begin{aligned}
\frac{\partial l_n}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[t_i(1-t_i)\right] - 2\sum_{i=1}^{n} \frac{t_i^{\alpha}\log(t_i) + (1-t_i)^{\alpha}\log(1-t_i)}{t_i^{\alpha} + (1-t_i)^{\alpha}},\\
\frac{\partial l_n}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{x_i^{\beta}\,\log(x_i)}{1+x_i^{\beta}} - \lambda \sum_{i=1}^{n} x_i^{\beta}\,\log(x_i)\\
&\quad +(\alpha-1)\sum_{i=1}^{n} \frac{t_i^{(\beta)}}{t_i} + (1-\alpha)\sum_{i=1}^{n} \frac{t_i^{(\beta)}}{1-t_i} - 2\alpha \sum_{i=1}^{n} t_i^{(\beta)}\, \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}},\\
\frac{\partial l_n}{\partial \lambda} &= \frac{2n}{\lambda} - \frac{n}{1+\lambda} - \sum_{i=1}^{n} x_i^{\beta} + (\alpha-1)\sum_{i=1}^{n} \frac{t_i^{(\lambda)}}{t_i} + (1-\alpha)\sum_{i=1}^{n} \frac{t_i^{(\lambda)}}{1-t_i}\\
&\quad -2\alpha \sum_{i=1}^{n} t_i^{(\lambda)}\, \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}},
\end{aligned}
$$

where

$$
\begin{aligned}
t_i^{(\beta)} &= \frac{\lambda^2}{1+\lambda} x_i^{\beta}(1+x_i^{\beta})\mathrm{e}^{-\lambda\, x_i^{\beta}}\,\log(x_i),\\
t_i^{(\lambda)} &= \frac{-x_i^{\beta}\,\mathrm{e}^{-\lambda x_i^{\beta}}}{(1+\lambda)^2} + x_i^{\beta}\big(1 + \frac{\lambda}{1+\lambda}\, x_i^{\beta}\big)\mathrm{e}^{-\lambda x_i^{\beta}}.
\end{aligned}
$$

For estimating the model parameters, numerical iterative techniques should be used to solve these equations. We can investigate the global maxima of the log-likelihood by setting different starting values for the parameters. The information matrix will be required for interval estimation. Let $\boldsymbol{\theta} = (\alpha, \beta, \lambda)^T$, then the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_3(0, K(\boldsymbol{\theta})^{-1})$, under standard regularity conditions (see Lehmann and Casella, 1998, pp. 461-463), where $K(\boldsymbol{\theta})$ is the expected information matrix. The asymptotic behavior remains valid if $K(\boldsymbol{\theta})$ is superseded by the observed information matrix multiplied by $\frac{1}{n}$, say $\frac{1}{n}I(\boldsymbol{\theta})$, approximated by $\widehat{\boldsymbol{\theta}}$, i.e. $\frac{1}{n}I(\widehat{\boldsymbol{\theta}})$. We have

$$
I(\boldsymbol{\theta}) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda}\\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\lambda}\\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} \end{bmatrix},
$$

where

$$
\begin{aligned}
I_{\alpha\alpha} &= \frac{\partial^2 l_n}{\partial \alpha^2} = \frac{-n}{\alpha^2} - 2\sum_{i=1}^{n} \frac{t_i^{\alpha}(1-t_i)^{\alpha}\log(t_i)\log(\frac{t_i}{1-t_i}) + t_i^{\alpha}(1-t_i)^{\alpha}\log(1-t_i)\log(\frac{1-t_i}{t_i})}{[t_i^{\alpha} + (1-t_i)^{\alpha}]^2},\\
I_{\alpha\beta} &= I_{\beta\alpha} = \frac{\partial^2 l_n}{\partial\alpha\partial\beta} = \sum_{i=1}^{n} \frac{t_i^{(\beta)}}{t_i} - \sum_{i=1}^{n} \frac{t_i^{(\beta)}}{1-t_i} - 2\sum_{i=1}^{n} t_i^{(\beta)}\, \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}}\\
&\quad - 2\alpha \sum_{i=1}^{n} \frac{t_i^{(\beta)} t_i^{\alpha-1}\,(1-t_i)^{\alpha-1}\,\log(\frac{t_i}{1-t_i})}{[t_i^{\alpha} + (1-t_i)^{\alpha}]^2},
\end{aligned}
$$

$$I_{\alpha\lambda} = I_{\lambda\alpha} = \frac{\partial^2 l_n}{\partial\alpha\partial\lambda} = \sum_{i=1}^{n} \frac{t_i^{(\lambda)}}{t_i} - \sum_{i=1}^{n} \frac{t_i^{(\lambda)}}{1-t_i} - 2\sum_{i=1}^{n} t_i^{(\lambda)} \; \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}}$$

$$- 2\alpha \sum_{i=1}^{n} t_i^{(\lambda)} \; \frac{t_i^{\alpha-1}\log(t_i) - (1-t_i)^{\alpha-1}\log(1-t_i)}{t_i^{\alpha} + (1-t_i)^{\alpha}}$$

$$+ 2\alpha \sum_{i=1}^{n} t_i^{(\lambda)} \; \frac{\left[t_i^{\alpha}\log(t_i) + (1-t_i)^{\alpha}\log(1-t_i)\right]\left[t_i^{\alpha-1} - (1-t_i)^{\alpha-1}\right]}{\left[t_i^{\alpha} + (1-t_i)^{\alpha}\right]^2},$$

$$I_{\beta\beta} = \frac{\partial^2 l_n}{\partial\beta^2} = \frac{-n}{\beta^2} + \sum_{i=1}^{n} x_i^{\beta} \left(\frac{\log(x_i)}{1+x_i^{\beta}}\right)^2 - \lambda \sum_{i=1}^{n} x_i^{\beta} \; [\log(x_i)]^2 + (\alpha-1)\sum_{i=1}^{n} \frac{t_i^{(\beta\beta)} t_i - \left[t_i^{(\beta)}\right]^2}{t_i^2}$$

$$+ (1-\alpha)\sum_{i=1}^{n} \frac{t_i^{(\beta\beta)}(1-t_i) + \left[t_i^{(\beta)}\right]^2}{(1-t_i)^2} - 2\alpha \sum_{i=1}^{n} t_i^{(\beta\beta)} \; \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}}$$

$$- 2\alpha(\alpha-1)\sum_{i=1}^{n} \left[t_i^{(\beta)}\right]^2 \; \frac{t_i^{\alpha-2} + (1-t_i)^{\alpha-2}}{t_i^{\alpha} + (1-t_i)^{\alpha}} + 2\alpha^2 \sum_{i=1}^{n} \left\{ t_i^{(\beta)} \; \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}} \right\}^2,$$

$$I_{\beta\lambda} = I_{\lambda\beta} = \frac{\partial^2 l_n}{\partial\beta\partial\lambda} = -\sum_{i=1}^{n} x_i^{\beta}\log(x_i) + (\alpha-1)\sum_{i=1}^{n} \frac{t_i^{(\beta\lambda)} t_i - t_i^{(\beta)} t_i^{(\lambda)}}{t_i^2}$$

$$+ (1-\alpha)\sum_{i=1}^{n} \frac{t_i^{(\beta\lambda)}(1-t_i) + t_i^{(\beta)} t_i^{(\lambda)}}{(1-t_i)^2} - 2\alpha \sum_{i=1}^{n} t_i^{(\beta\lambda)} \; \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}}$$

$$- 2\alpha(\alpha-1)\sum_{i=1}^{n} t_i^{(\beta)} t_i^{(\lambda)} \; \frac{t_i^{\alpha-2} + (1-t_i)^{\alpha-2}}{t_i^{\alpha} + (1-t_i)^{\alpha}} + 2\alpha^2 \sum_{i=1}^{n} t_i^{(\beta)} t_i^{(\lambda)} \left\{ \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}} \right\}^2,$$

$$I_{\lambda\lambda} = \frac{\partial^2 l_n}{\partial\lambda^2} = \frac{-2n}{\lambda^2} + \frac{n}{(1+\lambda)^2} + (\alpha-1)\sum_{i=1}^{n} \frac{t_i^{(\lambda\lambda)} t_i - \left[t_i^{(\lambda)}\right]^2}{t_i^2}$$

$$+ (1-\alpha)\sum_{i=1}^{n} \frac{t_i^{(\lambda\lambda)}(1-t_i) + \left[t_i^{(\lambda)}\right]^2}{(1-t_i)^2} - 2\alpha \sum_{i=1}^{n} t_i^{(\lambda\lambda)} \; \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}}$$

$$- 2\alpha(\alpha-1)\sum_{i=1}^{n} \left[t_i^{(\lambda)}\right]^2 \; \frac{t_i^{\alpha-2} + (1-t_i)^{\alpha-2}}{t_i^{\alpha} + (1-t_i)^{\alpha}} + 2\alpha^2 \sum_{i=1}^{n} \left\{ t_i^{(\lambda)} \; \frac{t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^{\alpha} + (1-t_i)^{\alpha}} \right\}^2,$$

in which

$$t_i^{(\beta\beta)} = \frac{\lambda^2}{1+\lambda} x_i^{\beta}\, \mathrm{e}^{-\lambda x_i^{\beta}} [\log(x_i)]^2 \left[1 + (2-\lambda)x_i^{\beta} - \lambda\, x_i^{2\beta}\right],$$

$$t_i^{(\beta\lambda)} = \frac{-\lambda\, x_i^{\beta}}{(1+\lambda)^2}\, \mathrm{e}^{-\lambda x_i^{\beta}} \; \log(x_i)(1+x_i^{\beta})[\lambda x_i^{\beta}(1+\lambda) - \lambda - 2],$$

$$t_i^{(\lambda\lambda)} = \frac{2x_i^{\beta}\, \mathrm{e}^{-\lambda x_i^{\beta}}}{(1+\lambda)^3} + \frac{2x_i^{2\beta}\, \mathrm{e}^{-\lambda x_i^{\beta}}}{(1+\lambda)^2} - x_i^{2\beta}(1 + \frac{\lambda}{1+\lambda}\, x_i^{\beta})\mathrm{e}^{-\lambda x_i^{\beta}}.$$

3.1 A simulation study

In order to assess the performance of the maximum likelihood method, a small simulation study is performed using the statistical software R. However, one can also perform in SAS by PROC NLMIXED procedure. The number of Monte Carlo replications was 30,000. For maximizing the log-likelihood function, one can use the MaxBFGS subroutine with analytical derivatives. The evaluation of the estimates was performed based on the following quantities for each sample size: the empirical mean squared errors (MSEs) are calculated using the R package from the Monte Carlo replications. The maximum likelihood (ML) estimates are determined for each simulated data, say, $(\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i)$ for $i = 1, 2, \cdots, 30,000$ and the biases and MSEs are computed by

$$bias_h(n) = \frac{1}{30000} \sum_{i=1}^{30000} (\hat{h_i} - h),$$

and

$$MSE_h(n) = \frac{1}{30000} \sum_{i=1}^{30000} (\hat{h_i} - h)^2,$$

for $h = \alpha, \beta, \lambda$. We consider the sample sizes at $n = 100, \quad 300$ and 500 and consider different values for the parameters. The empirical results are given in Table 1, and indicate that the estimates are quite stable and, more importantly, are close to the true values for these sample sizes. Furthermore, as the sample size increases, the MSEs decreases as expected.

Table 1. The biases and MSEs of the estimates under the maximum likelihood method.

Sample Size	Actual Value			Bias			MSE		
n	α	β	λ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
100	0.5	0.5	2	−0.1128	−0.0823	0.0386	0.0518	0.0460	0.0538
	0.5	0.5	3	−0.1319	0.0948	−0.1158	0.0180	0.0426	0.0978
	0.7	0.8	4	0.1281	−0.1231	−0.1734	0.0154	0.1207	0.1065
	0.9	0.7	6	0.1883	0.0937	−0.1019	0.0481	0.0222	0.0442
	1	1.5	0.9	0.1780	−0.0672	−0.0762	0.1372	0.0282	0.0917
	1.5	2	0.6	−0.0841	−0.1342	−0.1017	0.1273	0.0183	0.0737
300	0.5	0.5	2	.0717	0.0617	0.0361	0.0216	0.0230	0.0248
	0.5	0.5	3	0.1124	0.0853	0.0843	0.0084	0.0229	0.0457
	0.7	0.8	4	0.0838	0.1137	0.0505	0.0065	0.0591	0.0441
	0.9	0.7	6	0.1375	−0.0493	0.0811	0.0181	0.0105	0.0203
	1	1.5	0.9	0.1258	0.0563	0.0636	0.0646	0.0131	0.0281
	1.5	2	0.6	0.0343	0.0928	0.0429	0.1127	0.0089	0.0367
500	0.5	0.5	2	−0.0461	−0.0289	−0.0324	0.0094	0.0114	0.0111
	0.5	0.5	3	−0.0512	−0.1110	−0.0355	0.0041	0.0102	0.0229
	0.7	0.8	4	−0.0730	−0.0527	0.0467	0.0039	0.0315	0.0241
	0.9	0.7	6	−0.1023	−0.0208	−0.0786	0.0086	0.0053	0.0102
	1	1.5	0.9	−0.0783	−0.0425	−0.0169	0.0270	0.0071	0.0123
	1.5	2	0.6	0.0077	−0.0691	0.0326	0.0943	0.0040	0.0149

4. Application

In this section, we illustrate the power of OLL-PL distribution using two real data sets. The first data set, denoted as D1, refers to remission times in months of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003).

The second data set, referred as D2, includes the breaking times (in hours) for 76 Kevlar 49/epoxy strands that were subject to a stress of 373.9 ksi and a temperature of 110°C. The data were taken from Gómez et al. (2014) and they were previously reported by Glaser (1983).

For the purpose of comparison, we fitted the following models as well as the OLL-PL distribution to the above two data sets: (i) the Lindley distribution, (ii) the power Lindley (PL) distribution, (iii) the OLL-Lindley distribution, (iv) the EPL distribution, (v) the beta Lindley (BL) distribution whose pdf is given by (see Merovci and Sharma, 2014)

$$f_{BL}(x) = \frac{1}{B(\alpha,\beta)}[F_L(x)]^{\alpha-1}[1-F_L(x)]^{\beta-1}f_L(x), \quad x>0, \alpha, \beta, \lambda > 0,$$

where $f_L(\cdot)$ and $F_L(\cdot)$ are the pdf and cdf of the Lindley distribution with parameter λ, respectively and $B(a,b)$ is the complete beta function. and (vi) the generalized Lindley (GL) distribution, Nadarajah et al. (2011), a sub-model of beta Lindley distribution which will be obtained by setting $\beta = 1$.

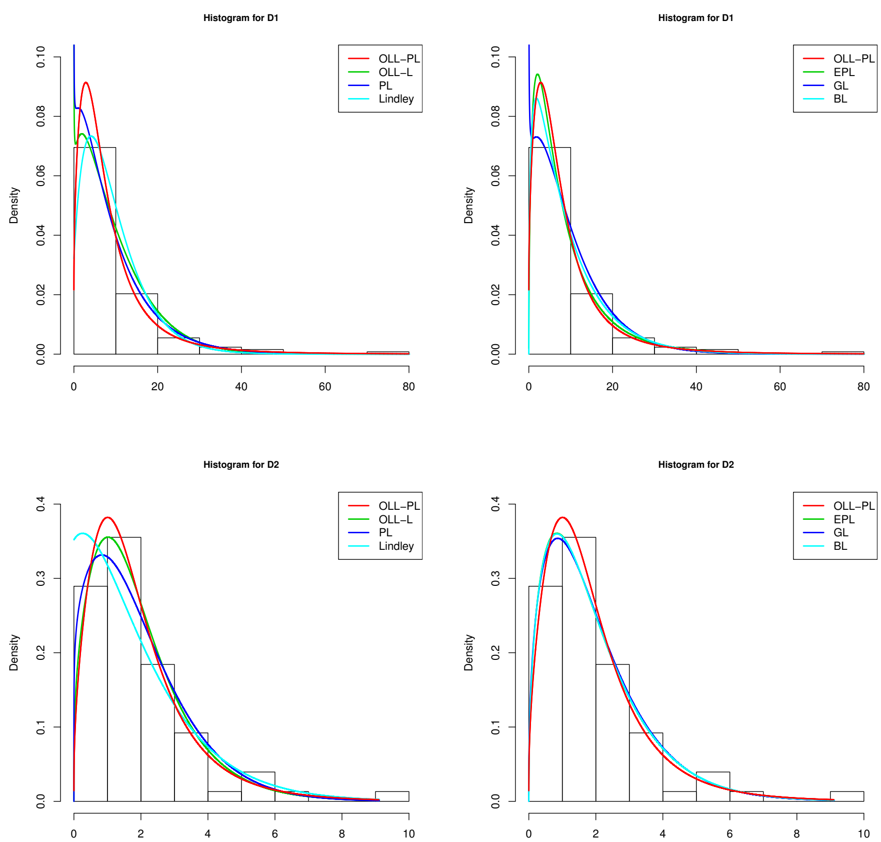

Figure 4. Histograms and the fitted pdfs for D1 and D2 (left plots for sub-models and right plots for the others).

We applied formal goodness-of-fit tests to determining which model fits the data best.

To this end, we considered the Cramér-von Mises (W^*) and Anderson-Darling (A^*) test statistics, see Chen and Balakrishnan (1995) for details regarding these statistics. Generally speaking, the smaller the values of A^* and W^*, the better the fit to the data. We have also considered the Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value and the minimum value of the minus log-likelihood function (-Log(L)) for the sake of comparison. The ML estimates of the parameters (standard errors in parentheses) as well as the goodness-of-fit test statistics for the two real data sets are presented in Table 2.

Table 2. Parameter ML estimates (standard errors in the parentheses) and the goodness-of-fit test statistics.

Results for D1

Model	α	β	λ	-log(L)	W^*	A^*	K-S	p-value
OLL-PL(α,β,λ)	2.5347 (1.295554)	0.4064 (0.18406)	0.60534 (0.17609)	409.4333	0.0157	0.1010	0.0316	0.9995
Lindley(λ)	1	1	0.1960 (0.012336)	419.5299	0.1717	1.0257	0.1164	0.0623
PL(β,λ)	1	0.8302 (0.047185)	0.29433 (0.03701)	413.3538	0.1177	0.7031	0.0682	0.5905
OLL-L(α,λ)	0.836265 (0.06481)	1	0.2032 (0.014994)	416.6386	0.1971	1.1770	0.1002	0.1529
EPL(α,β,λ)	2.76836 (1.290184)	0.5663 (0.101697)	0.8191 (0.311619)	410.4335	0.0392	0.2568	0.0429	0.9726
GL(α,λ)	0.73363 (0.09117)	1	0.16487 (0.016635)	416.2859	0.19205	1.1472	0.0928	0.2204
BL(α,β,λ)	1.34058 (0.431508)	0.0651 (0.056445)	1.8616 (1.465743)	412.8024	0.09997	0.6059	0.07136	0.5322

Results for D2

Model	α	β	λ	-log(L)	W^*	A^*	K-S	p-value
OLL-PL(α,β,λ)	1.7633 (0.70885)	0.7426 (0.26366)	0.8471 (0.105536)	120.8371	0.0773	0.4548	0.0818	0.6582
Lindley(λ)	1	1	0.7948 (0.067879)	123.6751	0.1173	0.6907	0.1156	0.2423
PL(β,λ)	1	1.1424 (0.090806)	0.7047 (0.081917)	122.4001	0.1289	0.7568	0.1123	0.2723
OLL-L(α,λ)	1.2592 (0.127996)	1	0.75064 (0.055503)	121.3641	0.0964	0.5699	0.0965	0.4507
EPL(α,β,λ)	1.5372 (0.666)	0.9497 (0.1937)	1.0213 (0.3596)	121.8663	0.1094	0.6469	0.0992	0.4156
GL(α,λ)	1.3903 (0.23753)	1	0.93626 (0.104684)	121.8991	0.11175	0.6594	0.1022	0.3795
BL(α,β,λ)	1.45537 (0.355)	0.7265 (0.835)	1.1942 (1.056)	121.8674	0.1097	0.6484	0.0998	0.4087

As we can see from Table 2, the smallest values of -Log(L), W^*, A^* and K-S statistics and the largest p-values belong to the OLL-PL distribution for both data sets. Therefore the OLL-PL distribution outperforms the other competitive considered distribution in the sense of these criteria.

For the first data set, the EPL distribution provides the second best fit and for the second data set, the OLL-L distribution provides the second best fit. These conclusions can also be drawn visually from Figures 4-6. Figures 5 and 6 reveal that the plotted points for the OLL-PL distribution best capture the diagonal line in the probability plots. Hence, the OLL-PL distribution could, in principle, be an appropriate model for fitting these data sets.

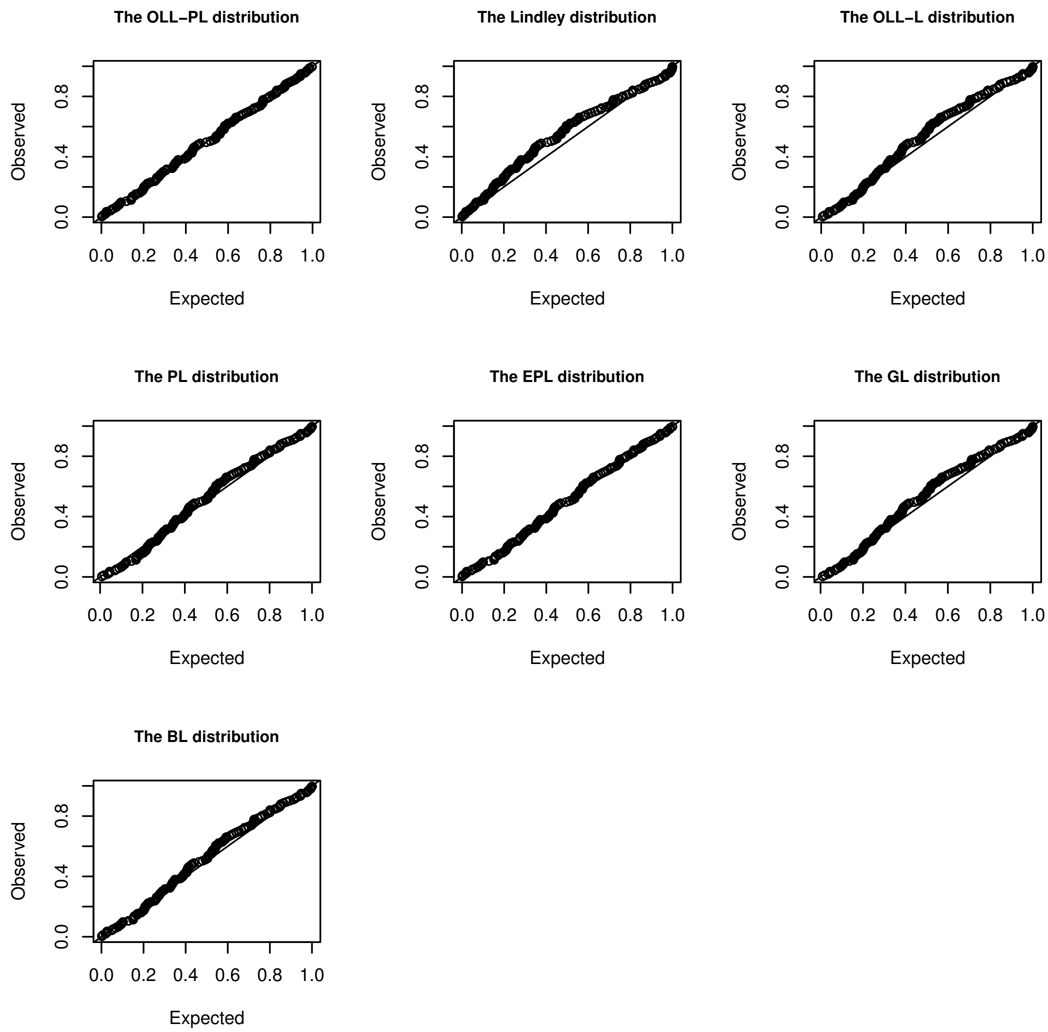

Figure 5. Probability plots for D1.

5. Concluding Remarks

In this paper, we introduce a new three-parameter distribution, so-called the odd log-logistic power Lindley (OLL-PL) distribution, that extends the generalized log logistic family of distributions by mixing with the Lindley distribution proposed by Lindley (1958). Several of its structural properties are discussed in detail. These include shape of the probability density function, hazard rate function and its shape, quantile function, limiting distributions of order statistics, and the general r-th moments. Moreover, the maximum likelihood estimation procedure is discussed for estimating the parameters. As can be seen from the shapes of the probability density and hazard functions, the new distribution provides more flexibility than other distributions that are commonly used for fitting lifetime data. A simulation study is also provided. Finally, two real-data examples are analyzed to show the applicability of the new distribution in practical situations. All the computations are performed using Maple 17 and R (R Core Team, 2016 and Marinho et al., 2013).

It is worthwhile to mention that other attractive properties of the new distribution are not considered in this paper, such as the reliability parameter, cumulants, cumulative residual entropy, distribution of the sum, product, difference & ratio of OLL-PL random variables, and bivariate & multivariate generalizations of the OLL-PL distribution, etc.

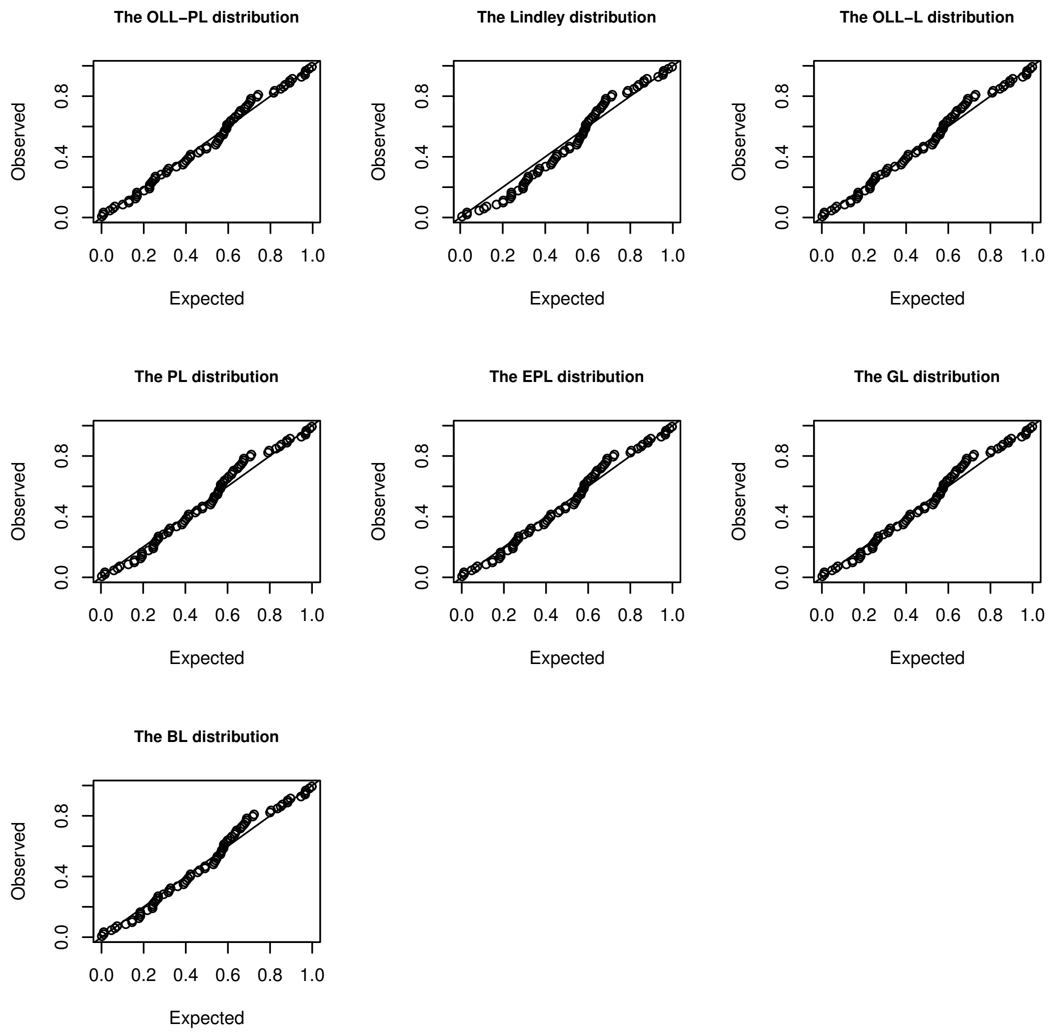

Figure 6. Probability plots for D2.

In an ongoing project, we plan to address some of these properties listed above. Bayesian estimates of the parameters are currently under investigation and will be reported elsewhere. Additionally, there are some open questions which we will try to address in future research:

- Is it possible to have characterizations of such families of distributions via order statistics?
- For the bivariate extension of the OLL-PL distribution and subsequently the multivariate extension, can we consider copula-based construction approach? If yes, then, what can we say regarding the dependence structure of the associated copula model(s)?
- Is it reasonable to consider the discrete analog of the continuous OLL-PL distribution?

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