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Research Paper

On the Hypothesis Testing for the Weighted Lindley Distribution

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Abstract

Recently, the two-parameter weighted Lindley distribution was proposed as a generalization for the one-parameter Lindley distribution. The proposed new distribution has an additional parameter leading to a more general form for the failure rate function. With appropriate choice of the parameter values, it is possible to model two aging classes of life distributions including bathtub and increasing hazard rates. It thus provides an alternative to many existing life distributions to modeling bathtub hazard rate. In this paper, based on a larger simulation experiment, we study the Type I error rate and power for the likelihood ratio, Wald, modified Wald, Score and Gradient tests used to distinguish the two-parameter weighted Lindley distribution from basic Lindley. With respect to size, under scenarios considered, the simulation study reveals that the likelihood ratio test performs better than the other ones. With respect to power, the Score test is found to perform better than the others.

Keywords: Hypothesis test · Lindley distribution · Type I error rate.

Mathematics Subject Classification: Primary 62-XX · Secondary 62Nxx.

1. Introduction

After Ghitany et al. (2008b), the one-parameter Lindley distribution has been generalized by several authors to increase its flexibility in the survival analysis data. One of these generalizations is the weighted Lindley which has as particular case the one-parameter Lindley distribution. According to Ghitany et al. (2011), a continuous and non-negative random variable T follows a two-parameter weighted Lindley distribution with shape parameters μ and β, both positive, if its probability density function is given by:

$$f(t \mid \mu, \beta) = \frac{\mu^{\beta+1}}{(\mu+\beta)\Gamma(\beta)} t^{\beta-1}(1+t)e^{-\mu t}, \tag{1}$$

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where $\Gamma(\beta)=\int_0^\infty t^{\beta-1}e^{-t}dt$ is the complete gamma function. From (1) we have the corresponding survival and hazard rate functions written, respectively, as:

$$S\left(t \mid \mu,\beta\right)=\frac{\left(\mu+\beta\right)\Gamma\left(\beta,\mu t\right)+\left(\mu t\right)^{\beta}e^{-\mu t}}{\left(\mu+\beta\right)\Gamma\left(\beta\right)}, \tag{2}$$

and

$$h\left(t \mid \mu,\beta\right)=\frac{\mu^{\beta+1}t^{\beta-1}\left(1+t\right)e^{-\mu t}}{\left(\mu+\beta\right)\Gamma\left(\beta,\mu t\right)+\left(\mu t\right)^{\beta}e^{-\mu t}}, \tag{3}$$

where $\Gamma\left(\beta,\mu t\right)$ is the upper incomplete Gamma function defined as $\int_{\mu t}^\infty t^{\beta-1}e^{-t}dt$ (Olver et al., 2010).

For $\beta=1$ in (1) we have the one-parameter Lindley probability density function as a particular case. The one-parameter Lindley distribution was introduced by Lindley (1965, 1958) as a distribution which can be useful to analyze lifetime data, especially in applications modeling stress-strength reliability. Ghitany et al. (2008b) studied the properties of the one-parameter Lindley distribution under a careful mathematical treatment. They also showed, in a numerical example, that the Lindley distribution presents better modeling than the one obtained by using the Exponential distribution. A generalized Lindley distribution, which includes as special cases the Exponential and Gamma distributions was proposed by Zakerzadeh and Dolati (2009). Ghitany and Al-Mutari (2008) considered a size-biased Poisson-Lindley distribution and Sankaran (1970) proposed the Poisson-Lindley distribution to model count data. Some properties of Poisson-Lindley distribution and its derived distributions were considered by Borah and Begum (2002). Borah and Deka Nath (2001a) considered the Poisson-Lindley and some of its mixture distributions. The zero-truncated Poisson-Lindley distribution and the generalized Poisson-Lindley distribution were considered by Ghitany et al. (2008a) and Mahmoudi and Zakerzadeh (2010), respectively. A study on the inflated Poisson-Lindley distribution was presented by Borah and Deka Nath (2001b) and Zamani and Ismail (2010) considered the negative binomial-Lindley distribution. The extended Lindley distribution was introduced by Bakouch et al. (2012) while Nadarajah et al. (2011) introduced the exponentiated Lindley distribution. The one-parameter Lindley distribution in the competing risks scenario was considered by Mazucheli and Achcar (2011). Several other generalizations can be found in the LindleyR R library (Mazucheli et al., 2016).

One nice feature of the two-parameter weighted Lindley distribution is that its hazard rate function is, for all $\mu>0$, bathtub shaped for $0<\beta<1$ and it increases for $\beta\geq 1$. Figure 1 shows the hazard rate function for the two-parameter weighted Lindley distribution for some different parameter values.

Since the standard one-parameter Lindley distribution does not provide enough flexibility to analyze different types of lifetime data, the two-parameter weighted Lindley distribution can be a good alternative lifetime distribution. The one-parameter Lindley distribution accommodates only increasing hazard rate (Ghitany et al., 2008b).

In the last years, several distributions have been proposed to model bathtub hazard rate behavior but in general these distributions have at least three parameters. Models with three or more parameters, considering limited amount of data may provide inaccurate estimates, so it is important to consider alternative models with small number of parameters. In addition to the weighted Lindley distribution, that can be used to model bathtub-shaped failure rate, three other distributions are also specified with two-parameters and are presented in Chen (2000), Haupt and Schäbe (1992) and in Smith and Bain (1975). A comprehensive review of the existing know distributions that exhibit bathtub shape is

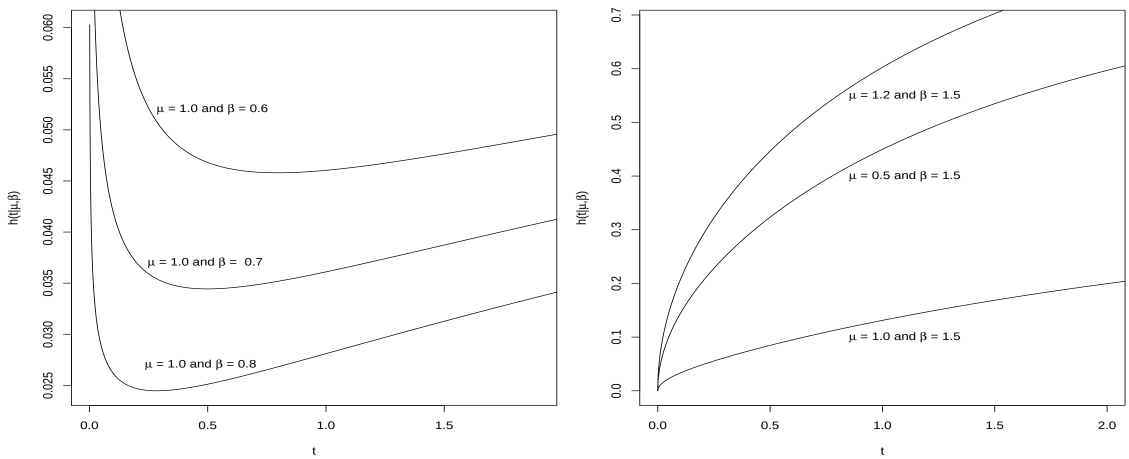


Figure 1. Behavior of the hazard rate function of weighted Lindley distribution.

provided by Rajarshi and Rajarshi (1988), Lai et al. (2001), Bebbington et al. (2007) and Nadarajah (2008).

In this paper it is conducted Monte Carlo simulation studies in order to evaluate the Type I error rate and power when the interest lies in discriminating the weighted Lindley distribution from the one-parameter Lindley distribution. The paper is organized as follows: Section 2 presents the likelihood function. The likelihood ratio test, Wald test, modified Wald test, Score test and Gradient test are presented in Section 3. Section 4 presents the specifications of the Monte Carlo simulation as well as the results obtained. Two applications are provided in Section 5. Final remarks in Section 6 ends up the paper.

2. The Likelihood Function

Let $\mathbf{t} = (t_1, \ldots, t_n)$ be a realization of the random sample $\mathbf{T} = (T_1, \ldots, T_n)$, where $T_1, \ldots, T_n$ are i.i.d. random variables according to the weighted Lindley distribution with parameter vector $\boldsymbol{\theta} = (\mu, \beta)$, $\mu > 0$ and $\beta > 0$. From (1) the likelihood function can be written as:

$$L(\boldsymbol{\theta} \mid \mathbf{t}) = \left(\frac{\mu^{\beta+1}}{(\mu+\beta)\,\Gamma(\beta)} \right)^n e^{-\mu t_0} \prod_{i=1}^n t_i^{\beta-1} (1+t_i), \tag{4}$$

where $t_0 = \sum_{i=1}^n t_i$ and $\Gamma(\beta)$ is the complete gamma function. From (4) the log-likelihood function for μ and β, $l(\boldsymbol{\theta} \mid \mathbf{t})$, is:

$$l(\boldsymbol{\theta} \mid \mathbf{t}) = n\left[(\beta+1)\log(\mu) - \log(\mu+\beta) - \log\Gamma(\beta)\right] - \mu t_0 + (\beta-1)\,t_1 + t_2, \tag{5}$$

where $t_1 = \sum_{i=1}^n \log(t_i)$ and $t_2 = \sum_{i=1}^n \log(1+t_i)$.

Differentiating (5) with respect to μ and β we have the Score vector $\boldsymbol{U}_\theta = \left[U_\mu \; U_\beta\right]$ with components:

$$U_\mu = n\left[\frac{\beta+1}{\mu} - \frac{1}{(\mu+\beta)}\right] - t_0, \tag{6}$$

and

$$U_{\beta}=n\left[\log\left(\mu\right)-\frac{1}{\left(\mu+\beta\right)}-\psi\left(\beta\right)\right]+t_{1},\tag{7}$$

where $U_{\mu}=\frac{\partial}{\partial\mu}l\left(\boldsymbol{\theta}\mid\mathbf{t}\right)$, $U_{\beta}=\frac{\partial}{\partial\beta}l\left(\boldsymbol{\theta}\mid\mathbf{t}\right)$ and $\psi(\beta)=d\log\Gamma(\beta)/d\beta=\Gamma'(\beta)/\Gamma(\beta)$ represent the digamma function.

To determine $\widehat{\mu}$ and $\widehat{\beta}$ we solve simultaneous equations $U_{\mu}=0$ and $U_{\beta}=0$ in μ and β, respectively. The equation $U_{\mu}=0$ can be solved algebraically for μ, giving:

$$\widehat{\mu}(\beta)=\frac{\beta(1-\overline{t})+\sqrt{\beta^{2}(1+\overline{t})^{2}+4\beta\overline{t}}}{2\overline{t}},\tag{8}$$

which is the maximum likelihood estimate of μ when β is assumed to be known, where $\overline{t}=t_{0}/n$. The maximum likelihood estimate for β can be obtained solving (7) by Newton's method or its modifications setting $\mu=\widehat{\mu}\left(\beta\right)$.

The maximum likelihood estimator of $\boldsymbol{\theta}$ can be considered as being approximately multivariate normal with mean $\boldsymbol{\theta}$ and a variance-covariance matrix that is the inverse of the expected information matrix. From Ghitany et al. (2011) the elements of the expected Fisher information are $I_{\mu\mu}=n\left(\frac{\beta+1}{\mu^{2}}-\frac{1}{(\mu+\beta)^{2}}\right)$, $I_{\mu\beta}=-n\left(\frac{1}{\mu}+\frac{1}{(\mu+\beta)^{2}}\right)$ and $I_{\beta\beta}=n\left(\psi'(\beta)-\frac{1}{(\mu+\beta)^{2}}\right)$ such that:

$$\boldsymbol{I_{\theta}}=n\begin{bmatrix}\frac{\beta+1}{\mu^{2}}-\frac{1}{(\mu+\beta)^{2}} & -\left[\frac{1}{\mu}+\frac{1}{(\mu+\beta)^{2}}\right]\\ -\left[\frac{1}{\mu}+\frac{1}{(\mu+\beta)^{2}}\right] & \psi'(\beta)-\frac{1}{(\mu+\beta)^{2}}\end{bmatrix},\tag{9}$$

where $\psi'\left(\beta\right)=\frac{d^{2}}{d\beta^{2}}\log\Gamma\left(\beta\right)$.

The inverse of $\boldsymbol{I_{\theta}}$ evaluated at $\widehat{\mu}$ and $\widehat{\beta}$ provides the asymptotic variance-covariance matrix of the maximum likelihood estimates. Since $\boldsymbol{I_{\theta}}$ is data independent it is equal to the observed information matrix.

3. Hypothesis Testing

Let us consider the two-parameter weighted Lindley distribution with the corresponding log-likelihood function $l\left(\boldsymbol{\theta}\right)$, Score vector $\boldsymbol{U_{\theta}}$ and Fisher information matrix $\boldsymbol{I_{\theta}}$ and suppose the interest is to test $H_{0}:\beta=1$ versus $H_{1}:\beta\neq1$ by treating μ as a nuisance parameter. In order to verify that the two-parameter weighted Lindley distribution is preferred, instead of the basic Lindley one, from which it is derived, we consider the following statistics: the likelihood ratio (S_{1}), the Wald-type (S_{2}), the modified Wald-type (S_{3}), the Score (S_{4}) and the Gradient (S_{5}) statistics as describe bellow.

The likelihood ratio test needs to maximize both, the restricted, $l_{R}\left(\mu,\beta=1\mid\mathbf{t}\right)$, and the unrestricted log-likelihood, $l_{U}\left(\mu,\beta\mid\mathbf{t}\right)$. Let $\widetilde{l}=l_{R}\left(\widetilde{\mu},\beta=1\mid\mathbf{t}\right)$ and $\widehat{l}=l_{U}(\widehat{\mu},\widehat{\beta}\mid\mathbf{t})$ the maximized log-likelihoods under H_{0} and H_{1}, respectively. From (8), $\widetilde{\mu}=\widehat{\mu}(1)$ and $\widehat{\mu}=\widehat{\mu}(\widehat{\beta})$ and under H_{0} the statistic:

$$S_{1}=2(\widehat{l}-\widetilde{l}),$$

is asymptotically distributed as chi-square distribution with one degree of freedom (Cox and Hinkley, 1974; Lehmann and Casella, 1998). From (5), the fitted restricted and unrestricted log-likelihood are, respectively:

$$\widetilde{l} = n\left[2\log\left(\widetilde{\mu}\right) - \log\left(\widetilde{\mu}+1\right)\right] - \widetilde{\mu}\, t_0 + t_2, \tag{10}$$

and

$$\widehat{l} = n[(\widehat{\beta}+1)\log(\widehat{\mu}) - \log(\widehat{\mu}+\widehat{\beta}) - \log\Gamma(\widehat{\beta})] - \widehat{\mu}\, t_0 + (\widehat{\beta}-1)t_1 + t_2. \tag{11}$$

The Wald-type test is based on the maximum likelihood estimates for μ and β, so it requires fitting just the unrestricted model. Since under the null hypothesis we have one restriction, the Wald-type statistic is written as:

$$S_2 = \frac{(\widehat{\beta}-1)^2}{\widehat{I}^{-1}_{\widehat{\beta}\widehat{\beta}}}, \tag{12}$$

where $\widehat{I}^{-1}_{\widehat{\beta}\widehat{\beta}} = \mathrm{var}(\widehat{\beta})$ is the element on 2^{nd} row and 2^{nd} column of the inverse matrix of $\boldsymbol{I_\theta}$, given in (9), evaluated at $\widehat{\mu}$ and $\widehat{\beta}$. If we consider the inverse matrix of $\boldsymbol{I_\theta}$ evaluated under H_0, that is at $\widetilde{\mu}$ and $\beta = 1$ we have the modified Wald-type (S_3) statistic, proposed by Hayakawa and Puri (1985). Under H_0, S_2 and S_3 are asymptotically distributed as a chi-square distribution with one degree of freedom.

The Score test is obtained by evaluating U_β and $\boldsymbol{I_\theta}$ under the null hypothesis, that is at $\widetilde{\mu}$ and $\beta = 1$. To test the hypothesis $H_0 : \beta = 1$ versus $H_1 : \beta \neq 1$ the Score statistic is defined as:

$$S_4 = U^2_{\widetilde{\beta}} \times \widetilde{I}^{-1}_{\widetilde{\beta}\widetilde{\beta}}, \tag{13}$$

where by (7) $U_{\widetilde{\beta}} = n\left[\log(\widetilde{\mu}) - \frac{1}{(\widetilde{\mu}+1)} + \psi(1)\right] + t_1$.

Such as the the likelihood ratio and the Wald-type statistics, under the null hypothesis, the Score statistic is also asymptotically distributed as chi-squared distribution with one degree of freedom. The Score test has an advantage over the likelihood ratio test and the Wald test in that the Score test only requires that the parameter of interest be estimated under the null hypothesis.

The Gradient test, recently proposed by Terrell (2002), shares the same first order asymptotic properties with the likelihood ratio, Wald and Score statistics. The Gradient statistic for testing $H_0 : \beta = 1$ versus $H_1 : \beta \neq 1$ is:

$$S_5 = (\widehat{\beta}-1) \times U_{\widetilde{\beta}}, \tag{14}$$

and asymptotically, under H_0 has a chi-square distribution with one degrees of freedom.

It is important to point out that these five tests are all asymptotically equivalent, but they may differ in finite samples (Buse, 1982). In practice, when these tests are evaluated numerically, they often lead to substantially different answers (Young and Smith, 2005). Under several scenarios, the Type I error rate and the power of the tests are compared in a simulation study in Section 4.

4. Simulation Study

In this section the tests introduced in the last section are compared with respect to the power and Type I error rate through a Monte Carlo study. For the power study the setup of the Monte Carlo experiments is as follows: the sample size was fixed at $n = 20$, 30, 50 and 100. The null hypothesis $H_0 : \beta = 1$ versus $H_1 : \beta \neq 1$ was tested generating n observations taking $\beta = 0.1$ to 2.0 by 0.01 and $\mu = 0.5$, 1.0 and 1.5. To simulate observations from the weighted Lindley distribution we have considered the procedure introduced in Ghitany et al. (2011), available in LindleyR R library (Mazucheli et al., 2016), and given by:

(1) Generate $u_1, \ldots, u_n$ from a $Uniform(0, 1)$,
(2) if $u_i \leq \frac{\theta}{\theta+\beta}$ generate x_i from a $Gamma(\beta, \theta)$ else generate x_i from a $Gamma(\beta + 1, \theta)$, $i = 1, \ldots, n$ (β and θ are the shape and scale parameters).

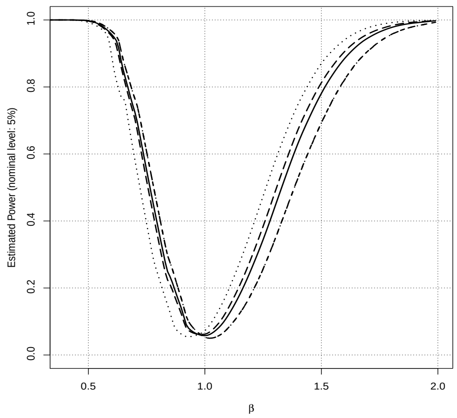
For each combination of n, β and μ simulation runs for $B = 100,000$ generated samples. The empirical power of the tests were obtained by calculating the proportion of times that S_j, $j = 1, \ldots, 5$, were greater than the critical value C, $C \cong 3.84\,(6.63)$ for nominal level $\alpha = 0.05\,(0.01)$.

All simulations were performed in *Ox* Console, version 6.20 (Doornik, 2007), using the *MaxBFGS* function to obtain the maximum likelihood estimates for μ and β whenever necessary. Below we describe the scheme used to estimate the Type I error rate. A similar scheme can be used to estimate the power.

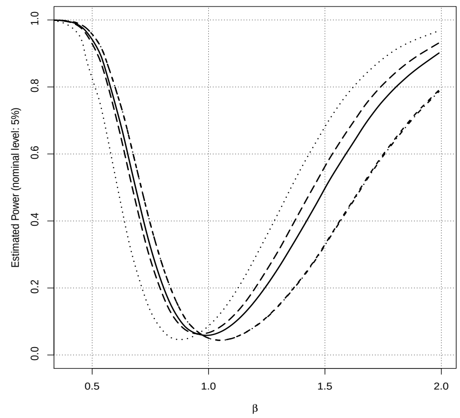
Let $I = 50$, $E = 0.00001$, $\beta = 1$, $\boldsymbol{\mu} = (0.5, 1.0, 1.5)$, $\boldsymbol{n} = 10(\text{by } 10), \ldots, 100$, and $B = 100,000$.

```
for i = 1, ..., length(μ) {
    μ = μ[i]
    for j = 1, ..., length(n)
    {
        n = n[j]; set k = 1;
        while k ≤ B
        {
            simulate (t_1, ..., t_n) using (Ghitany et al., 2011)
            estimate μ and β by the BFGS method
            if (I ≤ 50 or E ≤ 0.00001) then output μ̂ and β̂ else return
            k = k + 1
        }
        calculate S_1, ..., S_5
        calculate 1/B × #(S_i ≤ 0.05) and 1/B × #(S_i ≤ 0.01) for i = 1, ..., 5
}}
```

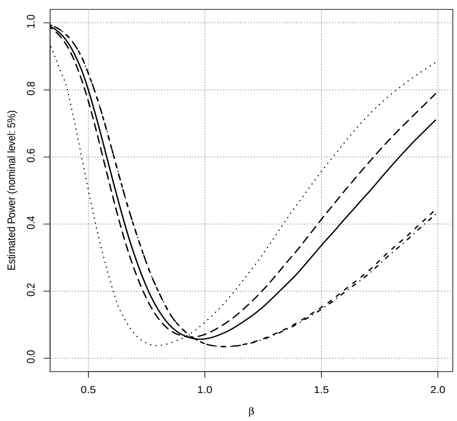
The estimated power for the five tests under different scenarios and $\mu = 0.5$ are shown in Figure 2. For the others values of μ we have observed a similar power behavior and the results are not shown.



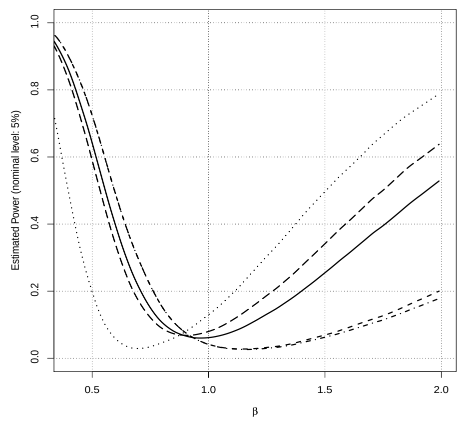
(a) $n = 100$



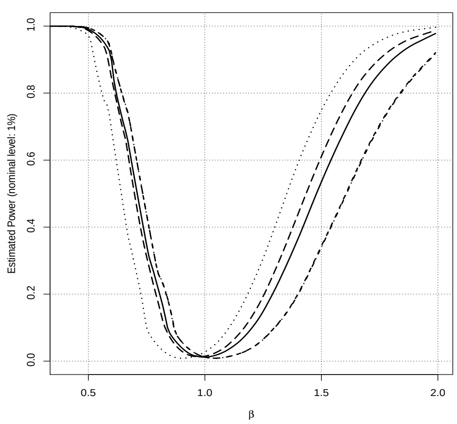
(b) $n = 50$



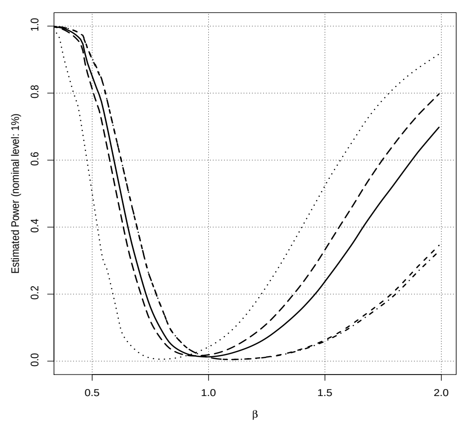
(c) $n = 30$



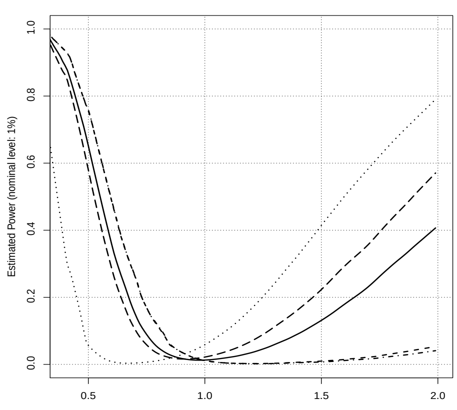
(d) $n = 20$



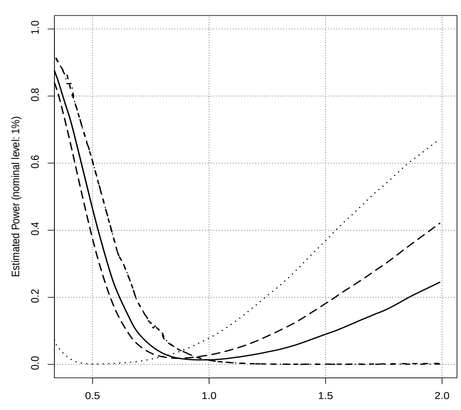
(e) $n = 100$



(f) $n = 50$



(g) $n = 30$

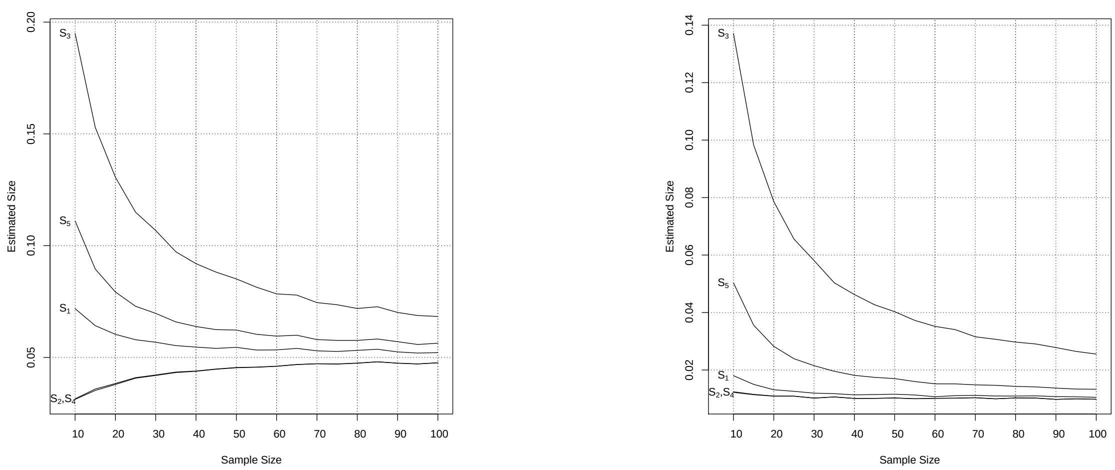


(h) $n = 20$

Figure 2. Estimated power (upper panels: nominal level = 5% and lower panels: nominal level = 1%) of the likelihood

Table 1. Area under estimated power curve.

		Statistical Tests									
		significance level: 5%					significance level: 1%				
μ	n	S_1	S_2	S_3	S_4	S_5	S_1	S_2	S_3	S_4	S_5
0.5	20	0.7400	0.5966	0.8013	0.5889	0.7981	0.5019	0.4578	0.5221	0.4577	0.5634
	30	0.8756	0.7407	0.9448	0.7350	0.9244	0.6169	0.5226	0.6853	0.5192	0.6790
	50	1.0720	0.9727	1.1289	0.9699	1.1054	0.8083	0.6674	0.9005	0.6618	0.8661
	100	1.3163	1.2767	1.3407	1.2761	1.3297	1.1145	1.0191	1.1694	1.0170	1.1457
1.0	20	0.6842	0.5598	0.7366	0.5456	0.7416	0.4617	0.4382	0.4617	0.4377	0.5148
	30	0.8082	0.6836	0.8746	0.6706	0.8583	0.5626	0.4979	0.6099	0.4932	0.6178
	50	0.9979	0.8920	1.0614	0.8839	1.0359	0.7322	0.6145	0.8216	0.6038	0.7902
	100	1.2505	1.2001	1.2838	1.1977	1.2683	1.0276	0.9239	1.0945	0.9163	1.0639
1.5	20	0.7677	0.6165	0.8281	0.6127	0.8243	0.5222	0.4646	0.5473	0.4647	0.5857
	30	0.9073	0.7707	0.9752	0.7685	0.9547	0.6428	0.5345	0.7161	0.5327	0.7069
	50	1.1041	1.0101	1.1551	1.0094	1.1342	0.8432	0.6983	0.9322	0.6957	0.8987
	100	1.3421	1.3084	1.3635	1.3083	1.3541	1.1518	1.0659	1.1998	1.0654	1.1797



(a) nominal level: 5% (b) nominal level: 1%

Figure 3. Estimated size of S_1 : Likelihood ratio test, S_2 : Wald test, S_3 : Modified Wald test, S_4 : Score test and S_5 : Gradient test.

Table 1 shows the area under estimated power curve calculated using the trapezoid rule integration. The larger the area the greater the power of the tests. By this criterion, for all n and μ considered, we have that the tests are ordered as: $S_3 > S_5 > S_1 > S_2 > S_4$. Figure 3 shows the estimated Type I error rate for the five tests, $n = 10$ to 100 by 1 and $\mu = 1$. We have observed similar Type I error rate for others values of μ and the Figure 3 indicate a difference in patterns behavior among these five tests. Here, also we have considered $B = 100,000$ generated samples.

5. Applications

In this section we fit the Lindley (L) and the weighted Lindley (WL) distribution to two real data sets and show the results of the applications of $S_1, \ldots, S_5$ hypothesis tests. The first data set was extracted from Jose et al. (2010), which refers to remission times (in months) of a random sample of 142 bladder cancer patients. The second data set was reported by Bjerkedal (1960), and employed by Gupta et al. (1997) among others. It represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, regimen 4.3. The regimen number is the common log of the number of bacillary units in 0.5 ml of challenge solution.

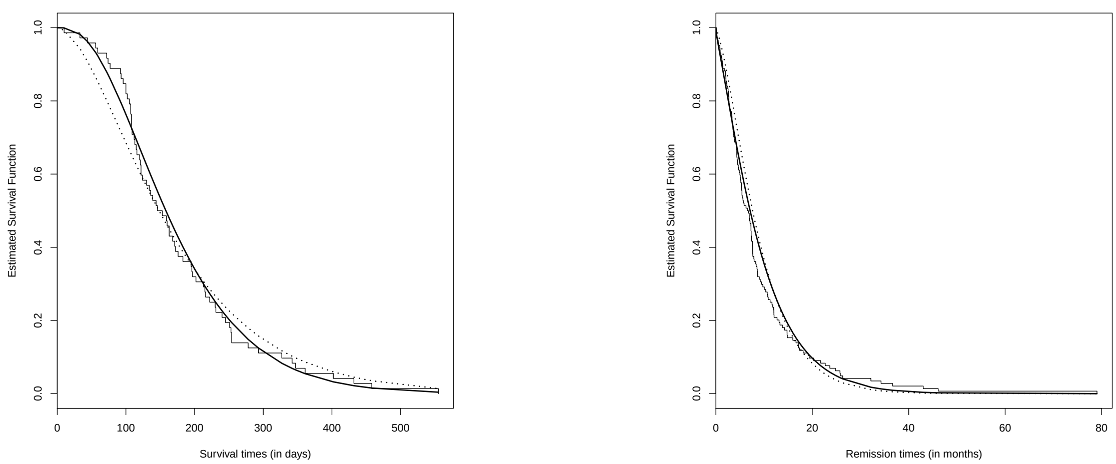
Table 2 list for both data sets and models (L) and (WL) the maximum likelihood estimates and their standard errors. The maximum likelihood estimates were obtained in *SAS/NLMIXED* procedure (SAS, 2010), by applying the Newton-Raphson algorithm. The results of the hypothesis testing are presented in Table 3 while the fitted survival curves are displayed in Figure 4.

Table 2. Maximum likelihood (standard error) estimates for Lindley and weighted Lindley distribution.

Model	Parameter	Data Set 1	Data Set 2
L	μ	0.2003 (0.0119)	0.0112 (0.0009)
WL	μ	0.1601 (0.0162)	0.0175 (0.0030)
	β	0.6603 (0.1010)	2.1052 (0.4872)

Table 3. Hypothesis testing results.

		Statistical Test				
		S_1	S_2	S_3	S_4	S_5
data set 1	Value	7.0085	5.1465	5.0948	13.7210	8.3608
	p-value	0.0038	0.0233	0.0240	0.0002	0.0038
data set 2	Value	8.4080	11.3172	11.1772	4.8680	7.3764
	p-value	0.0037	0.0008	0.0008	0.0274	0.0066



(a) data set 1 (b) data set 2

Figure 4. Plots of fitted Lindley (dotted line) and weighted Lindley (solid line) survival curves.

6. Concluding Remarks

In recent years, the Lindley distribution have been considered in several applications as an alternative lifetime model. Its generalization, the weighted Lindley distribution, is another alternative distribution to modeling lifetime data. In this sense, its important to have hypothesis testing procedures for distinguishes between both distributions. In these paper we considered hypothesis testing procedures based on likelihoods and their Type I error rate and power were studied by a large Monte Carlo study. The Monte Carlo study revealed with respect to size that, when the sample size increases, the Type I error rate of the tests approaches to the nominal level reasonably, but the likelihood ratio test performs better than the other ones, with this approximation being more slowly for S_5 test. With respect to power, the Score test was found to perform better than the others for small sample sizes. We fitted the weighted Lindley distribution to two real data sets and compared the obtained results with the one-parameter Lindley distribution in which showed the great potentialities of the WL distribution. The originality of this study comes from the fact that there has been no previous work comparing all of these hypothesis tests.

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