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Distribution Theory
Research Paper

Transmuted Dagum distribution with applications

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Abstract

In this article, we will use the quadratic rank transmutation map (QRTM) in order to generate a flexible family of probability distributions in which a three parameter Dagum distribution is embedded. Various structural properties of the new distribution including explicit expressions for the moments, random number generation and order statistics are derived. Estimation by maximum likelihood and inference for large samples are addressed. It will be shown that the analytical results are applicable to model the real world data.

Keywords: Dagum distribution · Transmutation map · Maximum likelihood · Reliability function · Order statistic.

Mathematics Subject Classification: Primary 60E05 · Secondary 62P99

1. Introduction

The quality of statistical analysis depends heavily on the underlying probability distribution. Because of this, considerable effort over the years has been expended in the development of large classes of probability distributions along with relevant statistical methodologies. In fact, the statistics literature is filled with hundreds of continuous univariate distributions but a real data set following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended forms of these distributions a significant progress has been made towards the generalization of some well-known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others.

In this article, we use the transmutation map method (Shaw and Buckley, 2009) to develop the so-called transmuted Dagum (TD) distribution by embedding a three parameter Dagum distribution. A random variable X is said to have a transmuted probability distribution with cdf $F(x)$ if

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \qquad |\lambda| \leq 1 \tag{1}$$

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where $G(x)$ is the cdf of the base distribution.

This generalization method has been deployed for generalization of many distributions by several authors including Aryal and Tsokos(2009, 2011), Aryal(2013), Ashour and Eltehiwy(2013a, 2013b), Elbatal(2013), Elbatal and Aryal(2013b), Elbatal and Elgarhy(2013c), Eltehiwy and Ashour (2013), Khan and King(2013), Mahmoud et al.(2013), Merovci(2013a, 2013b, 2013c), Tian et al.(2014) among others.

Dagum (1977) proposed the Dagum distribution to fit empirical income and wealth data, and also accommodate heavy tailed models. Dagum distribution has both Type-I and Type-II specification, where Type-I is the three parameter specifications and Type-II deals with four parameter specifications. Dagum motivated his model from empirical observation that the income elasticity of the cumulative distribution function (cdf) F of income into a decreasing and bounded function of F. In this context its features have been extensively analyzed by many authors, for excellent survey on the genesis and on empirical applications see (Kleiber et al.,2003) and (Kleiber, 2008). The cumulative distribution function (cdf) and probability density function (pdf) of Dagum (Type-I) distribution are given by

$$G(x, \alpha, \theta, \beta) = \left(1 + \alpha x^{-\theta}\right)^{-\beta}, \tag{2}$$

with $x > 0$,and $\alpha > 0, \theta > 0$ and $\beta > 0$,and

$$g(x, \alpha, \theta, \beta) = \alpha\theta\beta x^{-\theta-1}\left(1 + \alpha x^{-\theta}\right)^{-\beta-1}, \tag{3}$$

respectively, where α is a scale parameter, while θ and β are shape parameters. Throughout this paper, the Dagum distribution with parameters α, θ, and β will be denoted by $D(\alpha, \theta, \beta)$. The Dagum distribution has positive asymmetry, it is unimodal for $\theta\beta > 1$ and zero-modal for $\theta\beta \leq 1$. Also, an important characteristic of the Dagum distribution is that, according to the values of parameters, its hazard rate can be monotonically decreasing, upside-down bathtub and, finally, bathtub and then upside-down bathtub (Domma, 2002). This behavior has led several authors to study the model in different fields. In fact, recently, the Dagum distribution has been studied from a reliability point of view and used to analyze survival data see (Domma et al., 2011). Dagum (1980) refers to his model as the generalized logistic-Burr distribution. Actually when $\beta = 1$ Dagum distribution was also referred to as the log-logistic distribution. Also, generalized log- logistic distributions arise naturally in Burr's system of distributions. The most popular Burr distributions are Burr-XLL distribution, often called Burr distribution with cdf

$$F(x, \theta, \beta) = 1 - \left(1 + x^{-\theta}\right)^{-\beta}, \quad \text{for } x, \theta, \beta > 0$$

and the Burr-III distribution with cdf

$$F(x, \theta, \beta) = \left(1 + x^{-\theta}\right)^{-\beta}, \quad \text{for } x, \theta, \beta > 0.$$

It is clear that the Dagum distribution is a Burr III distribution with an additional scale parameter α.

The rest of the paper is organized as follows. In Section 2 we develop the expressions for pdf,cdf and reliability function of transmuted Dagum probability distribution. In Section 3 we studied the statistical properties include quantile functions, moments, moment generating function. The minimum , maximum and median order statistics models are discussed in Section 4. The Rényi and Shannon entropies are determined in Sections 5. In Section 6 we demonstrate the maximum likelihood estimates and the asymptotic confidence intervals of the unknown parameters. Finally, in Section 7 we present a real world data analysis to illustrate the usefulness of the proposed distribution.

2. Transmuted Dagum Distribution

In this section we develop the expressions for the pdf and cdf of transmuted Dagum (TD) distribution using the transmutation map. Using (1) and (2) we have the cdf of transmuted Dagum distribution

$$F_{TD}(x) = \left[\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\left[1+\lambda-\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right], \tag{4}$$

where θ and β are the shape parameters representing different patterns of the transmuted Dagum distribution and are positive, α is a scale parameter and λ is the transmuted parameter. The restrictions on the values of the parameters α, θ, β, and λ described in in equation (4) are always the same. The probability density function (pdf) of the transmuted Dagum distribution is given by

$$f_{TD}(x) = \alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-\beta-1}\left(1+\lambda-2\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right). \tag{5}$$

The transmuted Dagum distribution is very flexible model that approaches to different distributions when its parameters are changed. For example, for $\lambda = 0$ we have the Dagum distribution. Figure 1 illustrates the graphical behavior of the pdf of TD distribution for selected values of the parameters.

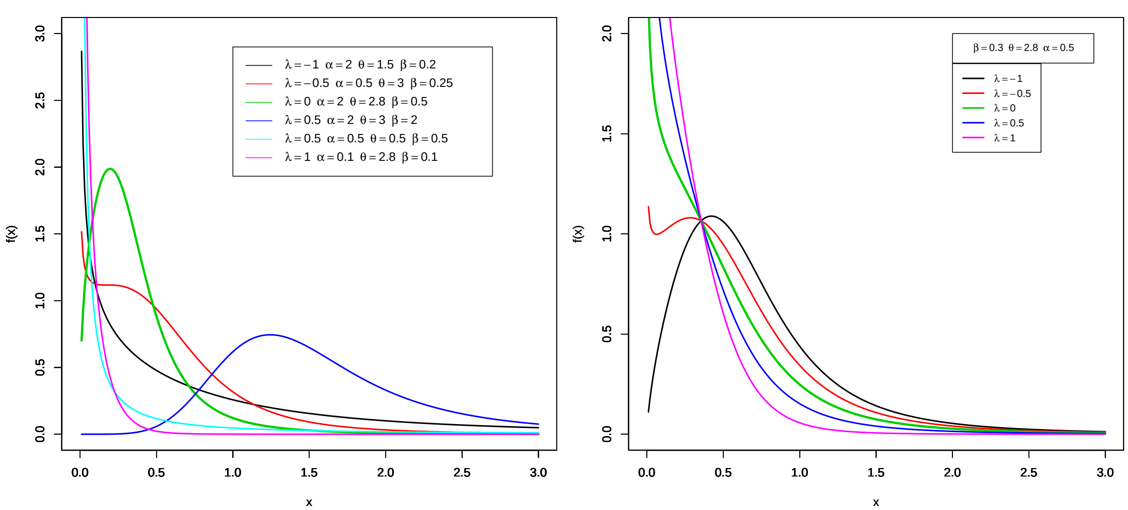

Figure 1. Probability density function of transmuted Dagum distribution for selected parameters

Note that the figure on the left exhibits the shape of the distribution for different choice of all the parameters whereas the figure on the right exhibits the behavior as λ varies from -1 to 1 while keeping all other three parameters fixed.
The cdf of transmuted Dagum distribution for selected values of the parameters are displayed in Figure 2.

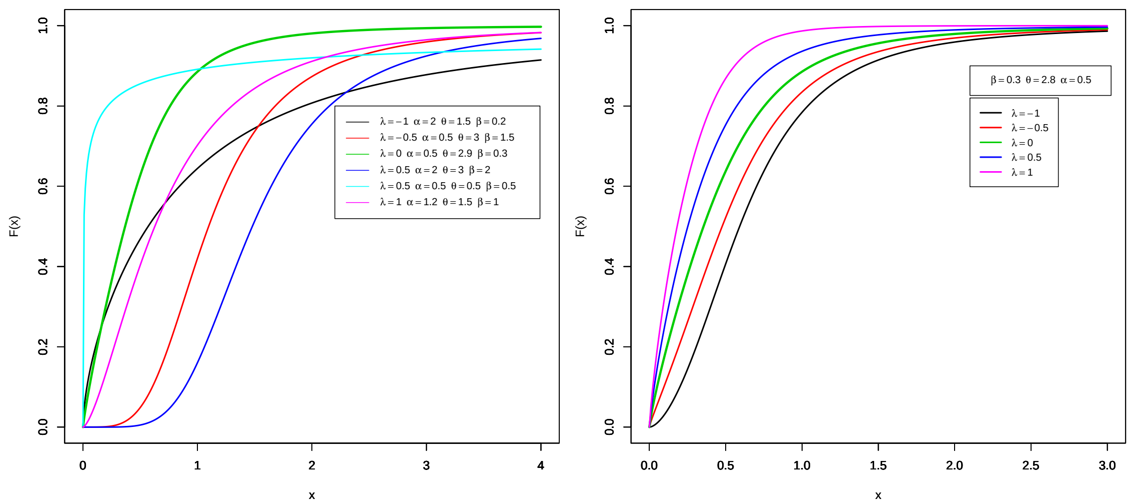

Figure 2. Cumulative distribution function of transmuted Dagum distribution for selected parameters

2.1 Reliability Analysis

The transmuted Dagum distribution can be a useful characterization of life time data analysis. The reliability function (RF) of the transmuted Dagum distribution is denoted by $R_{TD}(x)$ also known as the survivor function and is defined as

$$R_{TD}(x) = 1 - F_{TD}(x)$$

$$= 1 - \left[\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\left[1+\lambda-\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]. \quad (6)$$

It is important to note that $R_{TD}(x) + F_{TD}(x) = 1$. One of the characteristic in reliability analysis is the hazard rate function (hrf) defined by

$$h_{TD}(x) = \frac{f_{TD}(x)}{1-F_{TD}(x)}$$

$$= \frac{\alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-\beta-1}\left(1+\lambda-2\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right)}{1-\left[\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\left[1+\lambda-\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]}. \quad (7)$$

It is important to note that the units for $h_{TD}(x)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters.

Figure 3 illustrates the graphical behavior of the hazard rate function of the transmuted Dagum distribution for selected values of the parameters.

Note that the figure on the left displays the hazard rate function for different choice of all parameters whereas the figure on the right exhibits the behavior of the hazard rate function as λ varies from -1 to 1 while keeping all other three parameters fixed.

3. Statistical Properties

In this section we discuss the statistical properties of the transmuted Dagum distribution. Specifically quantile and random number generation function , moments and moment

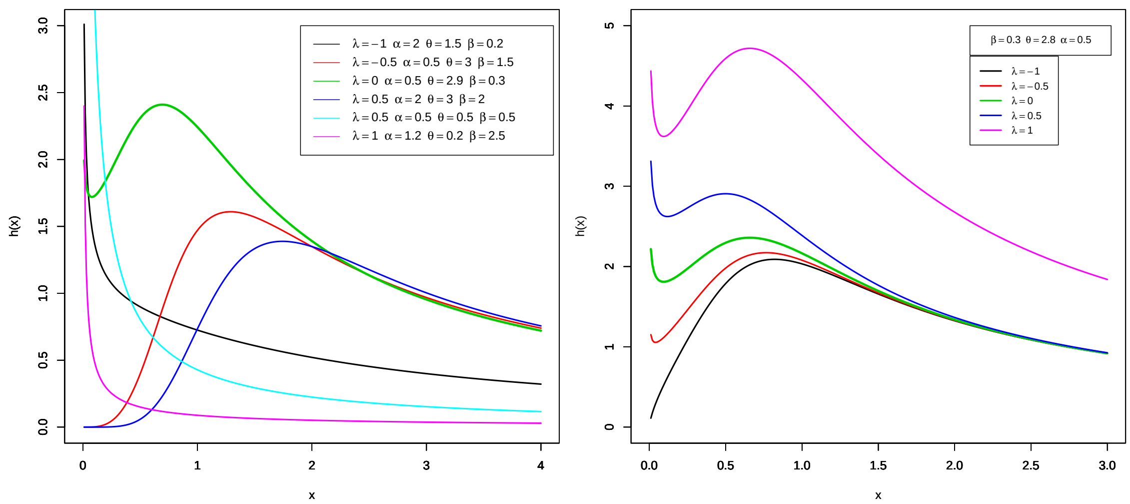

Figure 3. hazard rate function of transmuted Dagum distribution

generating function.

3.1 Quantile and Random Number Generation

The quantile x_q of the TD $(\alpha, \beta, \theta,\ \lambda, x)$ is real solution of the following equation

$$x_q = \alpha^{\frac{1}{\theta}} \left\{ -1 + \left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \right]^{\frac{-1}{\beta}} \right\}^{-\frac{1}{\theta}}. \tag{8}$$

In particular, the median of the transmuted Dagum distribution is

$$x_{0.5} = \alpha^{\frac{1}{\theta}} \left\{ -1 + 2^{\frac{1}{\beta}} \left[\frac{(1+\lambda) - \sqrt{1+\lambda^2}}{\lambda} \right]^{\frac{-1}{\beta}} \right\}^{-\frac{1}{\theta}}. \tag{9}$$

In order to generate random numbers from the TD distribution we can use the inversion method so we generate φ as uniform random variables from U(0,1) and use the relationship below.

$$x = \alpha^{\frac{1}{\theta}} \left\{ -1 + \left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda \varphi}}{2\lambda} \right]^{\frac{-1}{\beta}} \right\}^{-\frac{1}{\theta}}.$$

3.2 Moments & Moment Generating Function

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). If X has the TD $(\alpha, \beta, \theta, \lambda, x)$ then the r^{th} moment of X are given by the following

$$E(X^r) = \alpha^{\frac{r}{\theta}} \beta \left\{ (1+\lambda)\, B\left(\beta + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right) - 2\lambda B\left(2\beta + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right) \right\}, \tag{10}$$

where $B(.,.)$ is the beta function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Using the functional relationship between the beta and gamma function as given below

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

we have the r^{th} order moments are givable by

$$E(X^r)=\alpha^{\frac{r}{\theta}}\left\{(1+\lambda)\frac{\Gamma\left(\beta+\frac{r}{\theta}\right)\Gamma\left(1-\frac{r}{\theta}\right)}{\Gamma(\beta)}-2\lambda\frac{\Gamma\left(2\beta+\frac{r}{\theta}\right)\Gamma\left(1-\frac{r}{\theta}\right)}{2\Gamma(2\beta)}\right\}.$$

It should be noted that the gamma function is defined everywhere except negative integers and zero. Therefore the r^{th} order moment will be defined only when $\theta > r$. In particular,

$$E(X)=\begin{cases}\alpha^{\frac{1}{\theta}}\left[(1+\lambda)\frac{\Gamma\left(\beta+\frac{1}{\theta}\right)\Gamma\left(1-\frac{1}{\theta}\right)}{\Gamma(\beta)}-2\lambda\frac{\Gamma\left(2\beta+\frac{1}{\theta}\right)\Gamma\left(1-\frac{1}{\theta}\right)}{2\Gamma(2\beta)}\right], & \text{if } \theta>1,\\ \text{indeterminate,} & \text{Otherwise}\end{cases}$$

The variance, skewness and kurtosis of the TD distribution can be calculated from (10) using the relations given below.

$$\begin{aligned} Variance(X) &= E(X^2)-[E(X)]^2,\\ Skewness(X) &= \frac{E(X^3)-3E(X)E(X^2)+2E^3(X)}{Var^{3/2}(X)},\\ Kurtosis(X) &= \frac{E(X^4)-4E(X)E(X^3)+6E(X^2)E^2(X)-3E^4(X)}{Var^2(X)}.\end{aligned}$$

To illustrate the effect of the parameter λ on skewness and kurtosis we consider measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite, so it becomes uninformative. The Bowleys skewness (Kenney, 1962) is one of the earliest skewness measures defined by the average of the quartiles minus the median, divided by half the interquartile range, given by

$$\mathcal{B}=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}=\frac{Q(3/4)+Q(1/4)-2Q(2/4)}{Q(3/4)-Q(1/4)}$$

and the Moors kurtosis (Moors, 1998) is based on octiles and is given by

$$\mathcal{M}=\frac{(O_3-O_1)+(O_7-O_5)}{O_6-O_2}=\frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)}.$$

Figure 4 displays the Bowley ($\mathcal{B}$) and Moors ($\mathcal{M}$) kurtosis as a function of the parameter λ for $\alpha=2,\theta=10$ and $\beta=3$. It is evident that both measures depend on the parameter λ.

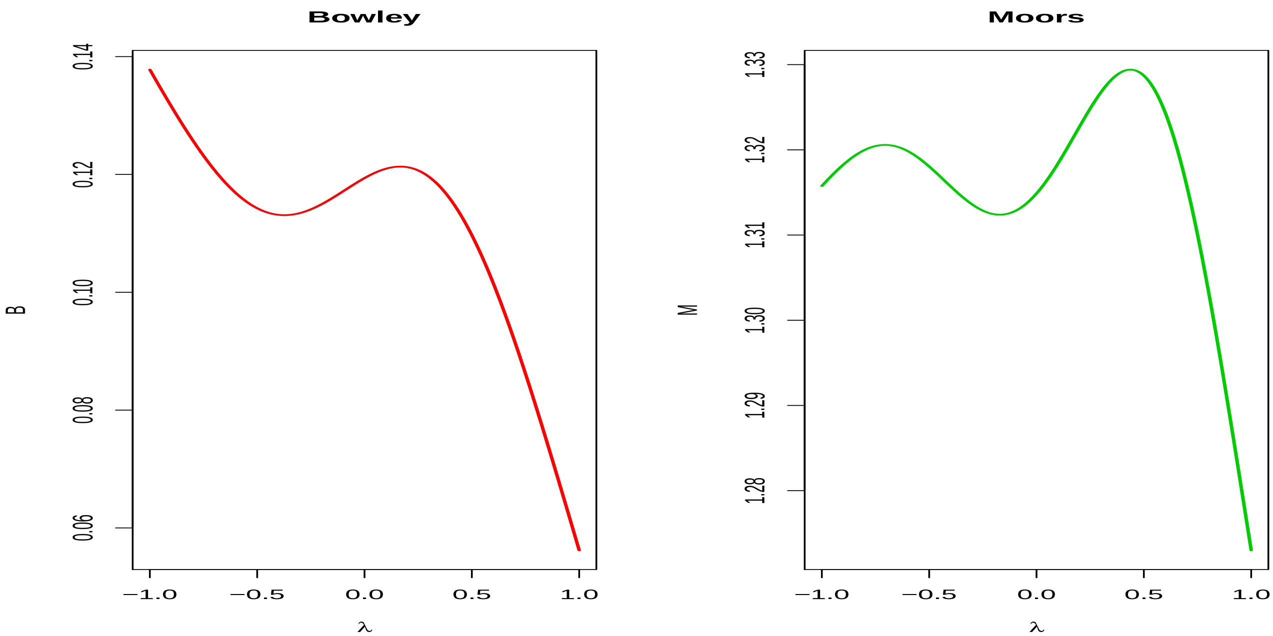

Figure 4. Behavior of Bowley(B) and Moors(M) kurtosis for transmuted Dagum distribution

The moment generating function, MGF, of a random variable X is defined by $M_X(t) = E(\exp(tX))$. When X has the TD $(\alpha, \beta, \theta, \lambda, x)$ then the MGF of X is given by

$$\begin{aligned} M_X(t) &= \int_0^\infty \exp(tx) f(x) dx \\ &= \sum\nolimits_{r=0}^{\infty} \frac{t^r}{r!} \alpha^{\frac{r}{\theta}} \beta \left\{ (1+\lambda)\, B\left(\beta + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right) - 2\lambda B\left(2\beta + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right) \right\} \end{aligned} \quad (11)$$

which can be expressed in terms of the gamma function as

$$M_X(t) = \sum\nolimits_{r=0}^{\infty} \frac{t^r}{r!} \alpha^{\frac{r}{\theta}} \left\{ (1+\lambda) \frac{\Gamma\left(\beta + \frac{r}{\theta}\right) \Gamma\left(1 - \frac{r}{\theta}\right)}{\Gamma(\beta)} - 2\lambda \frac{\Gamma\left(2\beta + \frac{r}{\theta}\right) \Gamma\left(1 - \frac{r}{\theta}\right)}{2\Gamma(2\beta)} \right\}.$$

As discussed above it should be noted that the MGF exists only if $\theta > r$.

4. Order Statistics

Let X_1, X_2,..., X_n be a simple random sample from TD distribution with cumulative distribution function and probability density function as in (4) and (5), respectively. Let $X_{(1:n)} \leq X_{(2:n)} \leq, \cdots, \leq X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i:n)}$ denote the lifetime of an $(n-i+1)-$out$-$of$-n$ system which consists of n independent and identically components. When $i = 1$, and when $i = n$, such systems are better known as series, and parallel systems, respectively. Considerable attention has been given to establish several reliability properties of such systems. It is well known that the cdf and pdf of $X_{(i:n)}$ for , $1 \leq i \leq n$ are, respectively, given by

$$F_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} [F(x)]^{k} [1 - F(x)]^{n-k}$$
$$= \int_{0}^{F(x)} \frac{n!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i} dt. \quad (12)$$

and

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x). \quad (13)$$

We define the smallest order statistic $X_{(1)} = Min(X_1, X_2, ..., X_n)$, the largest order statistic as $X_{(n)} = Max(X_1, X_2, ..., X_n)$ and median order statistic X_{m+1} where $m = \lfloor \frac{n}{2} \rfloor$.
Also note that in (13) $0 < F(x) < 1$ for $x > 0$, so by using the binomial series expansion we have

$$[1 - F(x)]^{n-i} = \sum_{k=0}^{n-i} (-1)^{k} \binom{n-i}{k} [F(x)]^{k}.$$

Therefore, the pdf of the order statistic (13) can be expressed as

$$f_{i:n}(x) = \sum_{k=0}^{n-i} (-1)^{k} \frac{n!}{(n-i-k)!(i-1)!k!} [F(x)]^{k+i-1} f(x)$$

Also note that the joint distribution of the the i^{th} and j^{th} order statistics for $1 \leq i < j \leq n$ is given by

$$f_{i:j:n}(x_i, x_j) = C [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-i-1} [1 - F(x_j)]^{n-j} f(x_i) f(x_j), \quad (14)$$

where $C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$.

4.1 Distribution of Minimum, Maximum and Median

Let $X_1, X_2, \cdots, X_n$ be a random sample from the transmuted Dagum distribution then the pdf the smallest order statistic $X_{(1)}$ is given by

$$f_{1:n}(x) = n [1 - F(x)]^{n-1} f(x)$$
$$= n\alpha\theta\beta x^{-\theta-1} \left(1 + \alpha x^{-\theta}\right)^{-\beta-1} \left\{1 + \lambda - 2\lambda \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right\}$$
$$\times \left\{1 - \left[\left(1 + \alpha x^{-\theta}\right)^{-\beta}\right] \left[1 + \lambda - \lambda \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right]\right\}^{n-1} \quad (15)$$

and the pdf of the largest order statistic $X_{(n)}$ is given by

$$
\begin{aligned}
f_{n:n}(x) &= n\left[F(x)\right]^{n-1} f(x)\\
&= n\alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-n\beta-1}\left\{1+\lambda-2\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right\}\\
&\quad\times\left\{1+\lambda-\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right\}^{n-1}. \qquad (16)
\end{aligned}
$$

and the pdf of median order statistic is

$$
\begin{aligned}
f_{m+1:n}(x) &= \frac{(2m+1)!}{m!m!}[F(x)]^m[1-F(x)]^m f(x)\\
&= \frac{(2m+1)!}{m!m!}\alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-(m\beta+\beta+1)}\left\{1+\lambda-2\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right\}\\
&\quad\times\left\{1+\lambda-\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right\}^m\left\{1-\left[\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\left[1+\lambda-\lambda\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\right\}^m.
\end{aligned}
$$

Similarly, using (14) the joint distribution of the the i_{th} and j_{th} order Statistics for $1 \leq i < j \leq n$ from transmuted Dagum distribution is

$$
\begin{aligned}
f_{i:j:n}(x_i,x_j) &= C\alpha^2\theta^2\beta^2\left[x_{(i)}x_{(j)}\right]^{-(\theta+1)}\left[h_{(i)}h_{(j)}\right]^{-(\beta+1)}\left\{\left[h_{(i)}^{-\beta}\right]\left[1+\lambda-\lambda h_{(i)}^{-\beta}\right]\right\}^{i-1}\\
&\quad\times\left\{1-\left[h_{(j)}^{-\beta}\right]\left[1+\lambda-\lambda h_{(j)}^{-\beta}\right]\right\}^{n-j}\left\{\left(1+\lambda-2\lambda h_{(i)}^{-\beta}\right)\left(1+\lambda-2\lambda h_{(j)}^{-\beta}\right)\right\}\\
&\quad\times\left\{\left[h_{(j)}^{-\beta}\right]\left[1+\lambda-\lambda h_{(j)}^{-\beta}\right]-\left[h_{(i)}^{-\beta}\right]\left[1+\lambda-\lambda h_{(i)}^{-\beta}\right]\right\}^{j-i-1},
\end{aligned}
$$

where, $h_{(i)} = \left(1+\alpha x_{(i)}^{-\theta}\right)$, $h_{(j)} = \left(1+\alpha x_{(j)}^{-\theta}\right)$ and $C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$. In a special case when $i = 1$ and $j = n$ we get the joint distribution of the minimum and maximum of order statistics and is given by

$$
\begin{aligned}
f_{1:n:n}(x_1,x_n) &= n(n-1)\ \left[F(x_{(n)})-F(x_{(1)})\right]^{n-2} f(x_{(1)})f(x_{(n)})\\
&= n(n-1)\alpha^2\theta^2\beta^2\left(x_{(1)}x_{(n)}\right)^{-(\theta+1)}\left(h_{(1)}h_{(n)}\right)^{-(\beta+1)}\left\{\left(1+\lambda-2\lambda h_{(1)}^{-\beta}\right)\left(1+\lambda-2\lambda h_{(n)}^{-\beta}\right)\right\}\\
&\quad\times\left\{\left[h_{(n)}^{-\beta}\right]\left[1+\lambda-\lambda h_{(n)}^{-\beta}\right]-\left[h_{(1)}^{-\beta}\right]\left[1+\lambda-\lambda h_{(1)}^{-\beta}\right]\right\}^{n-2},
\end{aligned}
$$

where, $h_{(1)} = \left(1+\alpha x_{(1)}^{-\theta}\right)$ and $h_{(n)} = \left(1+\alpha x_{(n)}^{-\theta}\right)$.

5. Entropy

In this section, we discuss the Rényi entropy and Shannon entropy for transmuted Dagum distribution. The concept of entropy plays a vital role in information theory (Rényi, 1961). The entropy of a random variable X is defined in terms of its probability distribution and can be shown to be a good measure of randomness or a measure of variation of the uncertainty. For a pdf $f(x)$, Rényi entropy is given by

$$H_R(f) = \frac{\log}{1-\delta}\int_0^\infty f^\delta(x)dx, \delta > 0, \delta \neq 1. \tag{17}$$

As $\delta \to 1$, we obtain the Shanon entropy. Note that for a transmuted Dagum distribution

$$\int_0^\infty f_{TD}^\delta(x)dx = \left\{[(1+\lambda)\,\alpha\theta\beta]^\delta \int_0^\infty x^{-\delta(\theta+1)}\left(1+\alpha x^{-\theta}\right)^{-\delta(\beta+1)} dx \right.$$
$$\left. - [2\lambda\alpha\theta\beta]^\delta \int_0^\infty \alpha\theta\beta x^{-\delta(\theta+1)}\left(1+\alpha x^{-\theta}\right)^{-\delta(2\beta+1)} dx\right\}.$$

Setting $t = \left(1+\alpha x^{-\theta}\right)^{-1}$ we get $x = (\alpha t)^{\frac{1}{\theta}}(1-t)^{\frac{-1}{\theta}}$ and we have

$$\int_0^\infty f_{TD}^\delta(x)dx = \frac{[(1+\lambda)\,\theta\beta]^\delta}{\theta\alpha^{\frac{\delta}{\theta}}} B\left(\beta\delta + \frac{1-\delta}{\theta}, \delta + \frac{\delta-1}{\theta}\right).$$

Therefore,

$$H_R(f) = \frac{1}{1-\delta}\log\left\{\frac{[(1+\lambda)\,\theta\beta]^\delta}{\theta\alpha^{\frac{\delta}{\theta}}} B(\beta\delta + \frac{1-\delta}{\theta}, \delta + \frac{\delta-1}{\theta})\right\}.$$

6. Estimation and Inference

In this section we will discuss about the method of parameter estimation of the transmuted Dagum distribution. The Maximum Likelihood Estimation is one of the most widely used estimation method for finding the unknown parameters. Here we find the estimators for the TD . Let $X_1, X_2, ..., X_n$ be a random sample from $X \sim TD(\alpha, \theta, \beta, \lambda)$ with observed values $x_1, x_2, ..., x_n$ then the likelihood function $L \equiv L(\alpha, \theta, \beta, \lambda : x_i)$can be written as

$$L = \prod_{i=1}^n \left\{\alpha\theta\beta x_i^{-\theta-1}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\right\}. \tag{18}$$

By taking logarithm of equation (18), the log-likelihood function $\ell = \ln L$ can be written as

$$\ell = n\ln\alpha + n\ln\theta + n\ln\beta - (\theta+1)\sum_{i=1}^n \ln x_i - (\beta+1)\sum_{i=1}^n \ln\left(1+\alpha x_i^{-\theta}\right)$$
$$+\sum_{i=1}^n \ln\left[1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right]. \tag{19}$$

The components of the score vector are obtained by differentiating (19) with respect to each parameter α, θ, β and λ as below:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - (\beta+1)\sum_{i=1}^{n} \frac{x_i^{-\theta}}{\left(1+\alpha x_i^{-\theta}\right)} + 2\lambda\beta \sum_{i=1}^{n} \frac{\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{x_i^{\theta}\left[1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right]},$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \ln x_i + \alpha(\beta+1)\sum_{i=1}^{n} \frac{\ln x_i}{x_i^{\theta}\left(1+\alpha x_i^{-\theta}\right)} - 2\alpha\lambda\beta \sum_{i=1}^{n} \frac{\ln x_i \, \left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{x_i^{\theta}\left[1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right]},$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \ln\left(1+\alpha x_i^{-\theta}\right) + 2\lambda \sum_{i=1}^{n} \frac{\left(1+\alpha x_i^{-\theta}\right)^{-\beta} \ln\left(1+\alpha x_i^{-\theta}\right)}{\left[1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right]},$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-2\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left[1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right]}.$$

The maximum likelihood estimators $\hat{\alpha}, \hat{\theta}, \hat{\beta}$ and $\hat{\lambda}$ of α, θ, β, and λ are obtained by setting the score vector to zero and solving the system of nonlinear equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi- Newton algorithm to numerically maximize the log-likelihood function given in (19). It should be noted that (19) can be written as

$$\ell(\boldsymbol{\phi}) = \sum_{i=1}^{n} \ell_i(\boldsymbol{\phi})$$

where, $\boldsymbol{\phi} = (\alpha, \theta, \beta, \lambda)^T$.

So, the score function takes the form

$$U(\boldsymbol{\phi}) = \frac{\partial \ell(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}} = \sum_{i=1}^{n} U_i(\boldsymbol{\phi}),$$

where, $U_i(\boldsymbol{\phi}) = \left(\frac{\partial \ell_i(\boldsymbol{\phi})}{\partial \alpha}, \frac{\partial \ell_i(\boldsymbol{\phi})}{\partial \theta}, \frac{\partial \ell_i(\boldsymbol{\phi})}{\partial \beta}, \frac{\partial \ell_i(\boldsymbol{\phi})}{\partial \lambda}\right)^T$ for $\quad i = 1, 2, \cdots, n$.

The Fisher information matrix, $\mathcal{I}(\boldsymbol{\phi})$, can be estimated by

$$\mathcal{I}(\hat{\boldsymbol{\phi}}) = \frac{1}{n}\sum_{i=1}^{n} U_i(\hat{\boldsymbol{\phi}}) U_i^T(\hat{\boldsymbol{\phi}}).$$

We can compute the maximum values of the unrestricted and restricted log-likelihood functions to obtain likelihood ratio (LR) statistics for testing the sub-model of the new distribution. For example, we can use the LR statistic to check whether the fitted transmuted Dagum distribution is statistically “superior” to a fitted Dagum distribution for a given data set. In this case we can compare the first model against the second model by testing $H_0 : \lambda = 0$ versus $H_a : \lambda \neq 0$.

7. Applications

In this section, we provide a data analysis in order to assess the goodness-of-fit of the model. The data set consists of the duration of time between successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes (Proschan, 1963). This data has been referenced by several authors including recently by Huang and Oluyede (2014). The descriptive summary of the data is provided in table 1.

Table 1. Descriptive statistics of the air conditioning

n	Mean	Median	Variance	Minimum	Maximum	Skewness	Kurtosis
188	92.0745	54	11645.93	1.00	603.00	2.139	8.023

We estimate the unknown parameters of the transmuted Dagum distribution using the method of maximum likelihood. There exists many maximization methods in R packages (The R Project). We will compute the maximum likelihood estimates (MLEs) using Limited- Memory quasi-Newton code for Bound-constrained optimization (L-BFGS-B). Table 2 lists the MLEs of the model parameters along with their errors provided in the he parentheses.

Table 2. Estimated Parameters for the air conditioning data

Model	α	θ	β	λ
TDD	1574.6018	1.6594	0.6749	0.1678
	(3913.96)	(0.3462)	(0.2718)	(0.5259)
DD*	94.1526	1.2626	1.2390	–
	(33.7549)	(0.0663)	(0.1749)	–

*Parameter estimates as of (Huang et al.)

The model selection is carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (consistent Akaike information criteria) and the HQIC (Hannan-Quinn information criterion). Note that the smaller the values of goodness-of-fit measures better the fit of the data. These measures are defined as

$$AIC = -2\ell(\hat{\boldsymbol{\phi}}) + 2q$$
$$BIC = -2\ell(\hat{\boldsymbol{\phi}}) + q\log(n)$$
$$HQIC = -2\ell(\hat{\boldsymbol{\phi}}) + 2q\log(\log(n))$$
$$CAIC = -2\ell(\hat{\boldsymbol{\phi}}) + \frac{2qn}{n-q-1}$$

where $\ell(\hat{\boldsymbol{\phi}})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters, and n is the sample size. Here $\boldsymbol{\phi}$ denotes the parameters. The AIC, BIC, HQIC and CAIC values for each model is provided in table 3.

Table 3. The AIC, BIC, HQIC and CAIC values for failure times

Model	$-\ell(\hat{\boldsymbol{\phi}})$	AIC	BIC	HQIC	CAIC
Transmuted Dagum	1037.05	2082.09	2095.04	2087.34	2082.31
Dagum	1039.20	2084.40	2094.11	2088.33	2084.53

Note that the LR test statistic for testing $H_0 : \lambda = 0$ versus $H_a : \lambda \neq 0$ is $\omega = 2(\ell(\hat{\boldsymbol{\phi}}) - \ell(\hat{\boldsymbol{\phi}}_0)) = 4.3$ with 1 degrees of freedom. Also note that $\chi^2_{0.05,1} = 3.84$. Therefore, at 5% level of significance we reject the null hypothesis and conclude that the subject data set can be modeled using the transmuted Dagum distribution.

The plots comparing the exact transmuted Dagum, Dagum and empirical cdf along with the P-P plot for this data is given in Figure 5. These plots indicate that the transmuted Dagum distribution fits the subject data well. Also note that for the subject data the Kolmogorov-Smirnov (KS) test statistic is $D = 0.0483$ with p-value$= 0.7732$. Similarly, Anderson-Darling (A) and Cramér-von Mises (W) statistic values are 0.596 and 0.083 respectively. These statistic values also support that the transmuted Dagum distribution fits the subject data well.

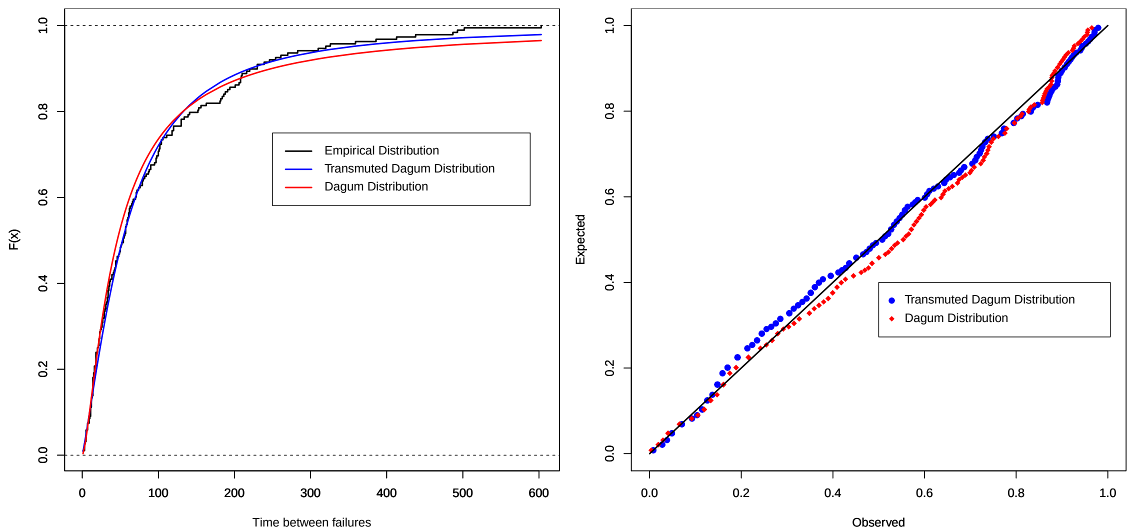

Figure 5. Fitted cdf and P-P plot of Dagum and transmuted Dagum distribution

8. Concluding Remarks

In the present study, we have introduced a new generalization of the Dagum distribution so-called the transmuted Dagum distribution. The 3-parameter Dagum distribution is embedded in the proposed distribution by introducing a shape parameter. Some mathematical properties along with estimation issues are addressed. We have presented an example where the transmuted Dagum distribution fits better than the Dagum distribution. We believe that the subject distribution can be used in several different areas. We also believe this study will serve as a reference to advance future research in the subject area.

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