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Burr Type-XII Percentile Control Charts

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Abstract

The Burr type XII has been studied for various applications of lifetime modelings. However, control charts related to the Burr type XII percentiles have not been seen in the literature. In this paper, three control charts for the Burr type XII percentiles are investigated. An extensive Monte Carlo simulation study is conducted to compare among the Shewhart-type chart, and parametric bootstrap charts that are respectively based on maximum likelihood estimator and a modified moment estimator by using in-control average run length and out-control average run length. Finally, an example is given for illustration.

Keywords: Average run length · Control charts · False alarm rate · Parametric bootstrap · Percentile · Shewhart chart.

Mathematics Subject Classification: Primary 62F40 · Secondary 62P30.

1. Introduction

The Burr type XII (BTXII) distribution was initially introduced by Burr (1942) as one of twelve distributions based on the differential equation $dF(x)/dx = F(x)(1 - F(x))g(x, F(x))$, where $g(x, y)$ is positive for $0 \leq y \leq 1$ and x is in the domain of $F(x)$. Since then, the BTXII distribution has received considerable attention in reliability study and failure time modeling due to its flexibility in shape. The cumulative distribution function (CDF) of the BTXII distribution can be defined as follows:

$$F(t; \alpha, \lambda) = 1 - (1 + t^{\lambda})^{-\alpha}; t > 0, \tag{1}$$

where $\alpha > 0$ and $\lambda > 0$ are shape parameters. Various aspects and properties of the BTXII distribution have been studied by many authors, for example, Al-Hussaini and Ali Mousa (1992), Wingo (1983), Wingo (1993), Wang and Keats (1996), Moore and Papadopoulos (2000), Chen and Yeh (2006), Lio et al (2010), and Lio and Tsai (2012). Tadikamalla (1980) investigated the connections of the BTXII distribution with some other distributions.

In many industrial applications, a specific quality condition of the product's lifetime is often required for engineering design consideration. Tadikamalla (1980) mentioned that the BTXII distribution can be used to fit almost any given unimodal lifetime data since

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it contains two shape parameters. However, to our best knowledge, no control charts for monitoring the BTXII lifetime percentiles have been presented in the literature. The well-known Shewhart-type control chart is constructed based on the assumption that data comes from a near-normal distribution. Although the sampling distribution of the maximum likelihood estimator of the BTXII percentile can be shown to be a normal distribution asymptotically, the exact sampling distribution of the maximum likelihood estimator of the BTXII percentile is unknown. In this case, the Shewhart-type control chart using a finite subgroup size may not provide appropriate control limits. Therefore, computer-based methods such as bootstrap methods could be good candidates to establish the control limits for monitoring BTXII percentiles. It is suggested to refer Gunter (1992), Efron and Tibsh (1993), and Young (1994) for comprehensive discussions of bootstrap techniques.

Bootstrap methods are helpful to establish control chart limits when the sampling distribution of a parameter estimator is not available. Many authors have studied the constructions of bootstrap charts. Bajgier (1992) developed a bootstrap chart to monitor the process mean, which was a competitor to the Shewhart $\overline{X}$ chart. However, if all pre-samples were not in control, the bootstrap chart could become conservative due to the fact that it produces too wide control limits, regardless of the underlying distribution of the process variable. Many referred papers, such as Liu and Tang (1996), Jones and Woodall (1998) and Seppala et al (1995), had pointed out that bootstrap charts could alarm for out-of-control status quicker than the Shewhart-type chart could if the underlying distribution of process variable was skewed. An advantage of bootstrap method is to release the restriction from the theoretical sampling distribution of an estimator. The computation time of a bootstrap method is perhaps a perceived disadvantage, but today's computer power has changed such perception. The advent of powerful and accessible computers has made any simulation-based process to be easily implemented and the computation results to be accomplished in an affordable amount of time.

Nichols and Padgett (2005) developed a parametric bootstrap chart (PBC) based on Weibull distribution for monitoring the tensile strength percentile in the production process of carbon fiber. They found out that the PBC could alarm for an out-of-control process quicker than the Shewhart-type chart, proposed by Padgett and Spurrier (1990). Lio and Park (2008) investigated PBCs for Birnbaum-Saunders percentiles based on maximum likelihood estimation method and moment method. From the simulation results, Lio and Park (2008) discovered that both bootstrap charts provided a shorter average run length(ARL) when the process was shifted to out-of-control. Lio and Park (2010) studied parametric bootstrap charts for inverse Gaussian percentiles and showed that the bootstrap charts performed better than the percentile control chart using Bonferroni bounds, which was provided by Onar and Padgett (2000). (Lio-Tsai-Aslam-Jiang-2014) showed that the bootstrap charts based on maximum likelihood estimation method and moment method performed better than the Shewhart-type chart for monitoring the Burr X percentiles. The bootstrap method uses bootstrap samples, which are generated by using a sample data of an estimator, to generate the sampling distribution of the estimator, and then provides appropriate control limits for a control chart. Only the usual conditions for a control chart setting, i.e. Phase I in-control pre-samples are available and subgroup observations are independent and identically distributed, are assumed.

In this article, a Shewhart-type chart and two PBCs, namely maximum likelihood estimation bootstrap (MLE-b) chart and modified moment estimation bootstrap (MME-b) chart, for monitoring the BTXII percentiles are studied. The rest of this paper is organized as follows: a brief introduction to the estimation methods of maximum likelihood and modified moment for the BTXII distribution parameters and percentiles are addressed in Section 2. Algorithms of building the Shewhart-type chart, the MLE-b chart and the MME-b chart for the BTXII percentiles are provided in Section 3. Intensive Monte Carlo simulations are conducted in Section 4 to evaluate the implementations of Shewhart-type, MLE-b, and MME-b charts for monitoring the BTXII percentiles. An example is presented in Section 5 for illustration and some conclusions are made in Section 6.

2. The burr type xii distribution

The BTXII distribution of (1) has probability density function (PDF) and percentile function, respectively, defined as:

$$f(t;\alpha,\lambda) = \lambda\alpha t^{\lambda-1}(1+t^{\lambda})^{-\alpha-1}; t > 0, \tag{2}$$

and

$$Q(p;\alpha,\lambda) = ((1-p)^{-1/\alpha} - 1)^{1/\lambda}; \qquad 0 < p < 1. \tag{3}$$

It can be easily shown that $Q(p;\alpha,\lambda)$ decreases with respect to α for a given value of p with $0 < p < 1$ and a value of λ with $\lambda > 0$; and for a given $0 < p < 1$, $Q(p;\alpha,\lambda)$ increases with respect to λ if $\alpha > -ln(1-p)/ln(2)$; otherwise, $Q(p;\alpha,\lambda)$ decreases with respect to λ. Let $\Theta^T = (\alpha,\lambda)$ and $\mathcal{T} = \{t_1, t_2, \ldots, t_n\}$ denote a size n random sample drawn from the BTXII distribution with PDF defined by Equation (2). Then the log-likelihood function can be presented as

$$L(\Theta) = nln(\alpha) + nln(\lambda) + (\lambda-1)\sum_{i=1}^{n} ln(t_i) - (\alpha+1)\sum_{i=1}^{n} ln(t_i^{\lambda}+1). \tag{4}$$

The maximum likelihood estimate (MLE), $\hat{\Theta}_n^T = (\hat{\alpha}_n, \hat{\lambda}_n)$, of Θ^T can be obtained by solving the following two nonlinear equations simultaneously,

$$\hat{\alpha}_n = \frac{n}{\sum_{i=1}^{n} ln(1.0 + t_i^{\hat{\lambda}_n})}, \tag{5}$$

$$\hat{\lambda}_n = \frac{n}{(\hat{\alpha}_n + 1)\sum_{i=1}^{n} \frac{t_i^{\hat{\lambda}_n} ln(t_i)}{t_i^{\hat{\lambda}_n}+1} - \sum_{i=1}^{n} ln(t_i)}. \tag{6}$$

Replacing Θ by $\hat{\Theta}_n$ in Equation (3), the MLE of the 100pth percentile is given as:

$$\hat{Q}_{p,n}(\hat{\Theta}_n) = ((1-p)^{-1/\hat{\alpha}_n} - 1)^{1/\hat{\lambda}_n}, \qquad 0 < p < 1. \tag{7}$$

The exact sampling distributions of $\hat{\alpha}_n$ and $\hat{\lambda}_n$ are not available, neither is the exact sampling distribution of $\hat{Q}_{p,n}(\hat{\Theta}_n)$. It can be shown that $\sqrt{n}(\hat{\Theta}_n - \Theta) \to N(\mathbf{0}, \mathbf{I}^{-1}(\Theta))$, where $\mathbf{0}$ is a two-dimensional column vector of zeros and $\mathbf{I}(\Theta)$ is the Fisher information matrix defined by

$$\mathbf{I}(\Theta) = \frac{-1}{n}\begin{bmatrix} E(\frac{\partial^2 L(\Theta)}{\partial\alpha^2}) & E(\frac{\partial^2 L(\Theta)}{\partial\alpha\partial\lambda}) \\ E(\frac{\partial^2 L(\Theta)}{\partial\lambda\partial\alpha}) & E(\frac{\partial^2 L(\Theta)}{\partial\lambda^2}) \end{bmatrix} = -\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \tag{8}$$

with

$$I_{11} = 1/\alpha^2,$$
$$I_{12} = I_{21} = \int_0^\infty \lambda\alpha(e^y - 1)^{1-1/\lambda} e^{-y(\alpha+1)} dy,$$
$$I_{22} = 1/\lambda^2 + (\alpha+1) \int_0^\infty \alpha ln(e^y - 1)(e^y - 1)^{1-1/\lambda} e^{-y(\alpha+2)} dy.$$

It can be shown that

$$\frac{\hat{Q}_{p,n}(\hat{\Theta}_n) - Q(p;\Theta)}{\sigma^2_{p,n}} \to N(0,1), \tag{9}$$

where

$$\sigma^2_{p,n} = \frac{1}{n} \nabla Q(p;\Theta)^T \mathbf{I}^{-1}(\Theta) \nabla Q(p;\Theta), \tag{10}$$

and $\nabla Q(p;\Theta)$ is the gradient of $Q(p;\Theta)$ with respect to Θ. Therefore, the Shewhart-type chart could be constructed, based on the asymptotic normal distribution, to monitor the BTXII percentile. Because the existence of I_{12} and I_{22} depends upon λ, and also the evaluations of I_{12} and I_{22} are difficult, the observed Fisher information matrix,

$$\hat{\mathbf{I}}_{\mathbf{n}}(\hat{\Theta}_n) = \frac{-1}{n} \begin{bmatrix} \frac{\partial^2 L(\Theta)}{\partial\alpha^2} & \frac{\partial^2 L(\Theta)}{\partial\alpha\partial\lambda} \\ \frac{\partial^2 L(\Theta)}{\partial\lambda\partial\alpha} & \frac{\partial^2 L(\Theta)}{\partial\lambda^2} \end{bmatrix}_{\Theta=\hat{\Theta}_n}, \tag{11}$$

without taking expectation, is used instead of $\mathbf{I}(\Theta)$. Denote ARL_0 and ARL_1 as in-control and out-of-control ARLs, respectively. In view of the simulation results reported in Section 4, it can be found that the simulated ARL_0 of the Shewhart-type chart seriously underestimates the corresponding nominal ARL_0. Hence, the Shewhart-type chart based on the MLE of $\hat{Q}_{p,n}$ will not be recommended to monitor BTXII percentiles in practice.

Let T be the BTXII distribution random variable and $\Gamma(x)$ be the gamma function. Given a positive integer s, the sth moment for BTXII distribution can be proved to be

$$E(T^s) = \alpha B(s/\lambda + 1, \alpha - s/\lambda), \quad \text{where} \quad \alpha > s/\lambda \quad \text{and} \quad B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y). \tag{12}$$

Hence, by equating the first two sample moments to the corresponding population moments, the following equations can be used to find moment method estimates (MMEs),

$$E(T) = \sum_{i=1}^n t_i/n \quad \text{and} \quad E(T^2) = \sum_{i=1}^n t_i^2/n. \tag{13}$$

Unfortunately, the procedure to solve Equation (13) can not be easily developed. Notice that the median of the BTXII can be shown to be $(2^{1/\alpha} - 1)^{1/\lambda}$. By letting sample median, t_{med}, be population median and sample mean be $E(T)$, the solution to the following equations,

$$\tilde{\alpha} = \frac{ln(2)}{ln(t_{med}^{\tilde{\lambda}} + 1.0)} \quad \text{and} \quad E(T) = \sum_{i=1}^n t_i/n, \tag{14}$$

seems easier to find. However, from the simulation experience, the solution of Equation (14) is still difficult to obtain for small sample size.

Let $t_{(1)} < t_{(2)} < \cdots < t_{(n)}$ be the order statistics of $\mathcal{T} = \{t_1, t_2, \ldots, t_n\}$. It can be shown that $\alpha ln(1+t_{(1)}^{\lambda}), \alpha ln(1+t_{(2)}^{\lambda}), \cdots, \alpha ln(1+t_{(n)}^{\lambda})$ is a random sample from the exponential distribution with mean one. Denote

$$X_1 = n\alpha ln(1 + t_{(1)}^{\lambda}), \tag{15}$$

$$X_2 = (n-1)\left(\alpha ln(1 + t_{(2)}^{\lambda}) - \alpha ln(1 + t_{(1)}^{\lambda})\right), \tag{16}$$

$$X_3 = (n-2)\left(\alpha ln(1 + t_{(3)}^{\lambda}) - \alpha ln(1 + t_{(2)}^{\lambda})\right), \tag{17}$$

$$\vdots \tag{18}$$

$$X_n = \left(\alpha ln(1 + t_{(n)}^{\lambda}) - \alpha ln(1 + t_{(n-1)}^{\lambda})\right). \tag{19}$$

Then $X_1, X_2, \cdots, X_n$ is a random sample from the exponential distribution with mean one. Let $g(\lambda) = 2\sum_{i=1}^{n-1}(ln(T_n) - ln(T_i))$, where $T_i = \sum_{j=1}^{i} X_j/\alpha$. It can be shown that $g(\lambda)$ has chi-square distribution with degree of freedom of $2n-2$ and αT_n has gamma distribution $G(1, n)$. Following the same argument of Wang (2008), λ can be estimated by the unique solution of $g(\widetilde{\lambda}) = 2(n-2)$ and α can be estimated by $\widetilde{\alpha} = (n-1)/T_n$. The estimates $\widetilde{\lambda}$ and $\widetilde{\alpha}$, which are called modified moment-method estimates (MMEs) of λ and α respectively, are developed based on the first order population moment with small sample size adjustment. Then the BTXII percentile $Q(p; \Theta)$, based on MME, $\tilde{\Theta}_n$, can be computed and denoted by $\tilde{Q}_{p,n}(\tilde{\Theta}_n)$. However, the exact sampling distributions of $\tilde{\Theta}_n$ and $\tilde{Q}_{p,n}(\tilde{\Theta}_n)$ are not available.

3. The Shewhart-type and parametric bootstrap charts

In Phase I, it is assumed that k in-control pre-samples of each size m are drawn from the BTXII distribution of (1) for the control chart setting. Let $n = m \times k$ denote the total sample size used in Phase I. A Shewhart-type chart and two PBCs are constructed in the following subsections.

3.1 Shewhart-type Chart

Through the MLE estimation procedure described in Section 2, the MLE of the 100pth percentile, using a size m sample from a Phase I in-control process, can be obtained via $\hat{Q}_{p,m}(\hat{\Theta}_m) = ((1.0-p)^{-1/\hat{\alpha}_m} - 1)^{1/\hat{\lambda}_m}$, where $\hat{\Theta}_m = (\hat{\alpha}_m, \hat{\lambda}_m)$ is the MLE of $\Theta = (\alpha, \lambda)$. Then, the Shewhart-type chart for monitoring the 100pth percentile, $Q(p; \Theta)$, can be constructed as follows:

(1) Using n sample observations from Phase I in-control process, the MLE, $\hat{\Theta}_n^T = (\hat{\alpha}_n, \hat{\lambda}_n)$, via formulas (5) and (6), is obtained and the asymptotic standard error of $\hat{Q}_{p,m}(\hat{\Theta}_m)$ can be estimated by

$$\text{SE}_{Q_m} = \sqrt{\frac{1}{m}\nabla Q^T(p; \hat{\Theta}_n)\hat{\mathbf{I}}_n(\hat{\Theta}_n)\nabla Q(p; \hat{\Theta}_n)}. \tag{20}$$

(2) For the jth pre-sample of size m, the MLE of $Q(p; \Theta)$ is obtained using formulas

(5), (6) and (7) and is denoted by $\hat{Q}^j_{p,m}(\hat{\Theta}^j_m)$, for $j = 1, 2, \ldots, k$. The sample mean of $\hat{Q}^j_{p,m}(\hat{\Theta}^j_m)$, $j = 1, 2, \ldots, k$, is computed and labeled as

$$\bar{\hat{Q}}_{p,m}(\hat{\Theta}_m) = \frac{1}{k}\sum_{j=1}^{k}\hat{Q}^j_{p,m}(\hat{\Theta}^j_m). \tag{21}$$

(3) The control limits of the Shewhart-type chart are presented as follows:

$$\mathrm{UCL}_{\mathrm{SH}} = \bar{\hat{Q}}_{p,m}(\hat{\Theta}_m) + z_{(1-\gamma/2)} \times \mathrm{SE}_{Q_m}, \tag{22}$$

$$\mathrm{LCL}_{\mathrm{SH}} = \bar{\hat{Q}}_{p,m}(\hat{\Theta}_m) - z_{(1-\gamma/2)} \times \mathrm{SE}_{Q_m}, \tag{23}$$

and the center line (CL) is $\mathrm{CL}_{\mathrm{SH}} = \bar{\hat{Q}}_{p,m}(\hat{\Theta}_m)$, where $z_{1-\gamma/2}$ satisfies $\Phi(z_{1-\gamma/2}) = 1 - \gamma/2$, with $0 < \gamma < 1$; $\Phi(\cdot)$ is the standard normal CDF and γ is called a false alarm rate (FAR).

After the control limits of the Shewhart-type chart are determined based on Phase I in-control samples, future samples of each size m (Phase II samples) are drawn from the BTXII process to compute the plot statistic $\hat{Q}_{p,m}(\hat{\Theta}_m)$. If $\hat{Q}_{p,m}(\hat{\Theta}_m)$ is plotted between control limits, $\mathrm{LCL}_{\mathrm{SH}}$ and $\mathrm{UCL}_{\mathrm{SH}}$, then the process is assumed to be in control. Otherwise, signal the process out-of-control.

3.2 Bootstrap Charts

The PBC based on MLE for monitoring BTXII percentiles is constructed by the following steps:

(1) Use n sample observations from Phase I in-control process to obtain the MLE, $\hat{\Theta}^T_n = (\hat{\alpha}_n, \hat{\lambda}_n)$, via formulas (5) and (6).
(2) Generate m parametric bootstrap observations from the BTXII distribution of (1) but replacing α and λ by the corresponding MLEs, $\hat{\alpha}_n$ and $\hat{\lambda}_n$, obtained from Step (1). Denote these parametric bootstrap observations by $x^*_1, x^*_2 \ldots, x^*_m$.
(3) Find the MLEs of α and λ using parametric bootstrap observations, $x^*_1, x^*_2 \ldots, x^*_m$, and denote the obtained MLEs by $\hat{\alpha}^*_m$ and $\hat{\lambda}^*_m$, respectively.
(4) Compute the bootstrap estimate of the 100pth percentile according to the formula:

$$\hat{q}^*_p = \hat{Q}^*_{p,m}(\hat{\Theta}) = ((1.0 - p)^{-1/\hat{\alpha}^*_m} - 1)^{1/\hat{\lambda}^*_m}. \tag{24}$$

(5) Repeat Step (2) to Step (4) B times to obtain a size B bootstrap sample, $\hat{q}^*_{p,1}, \hat{q}^*_{p,2}, \ldots, \hat{q}^*_{p,B}$, where B is a given large positive integer.
(6) Given a FAR, γ, find the $(\gamma/2)$th and $(1-\gamma/2)$th empirical quantiles of the bootstrap sample, $\hat{q}^*_{p,1}, \hat{q}^*_{p,2}, \ldots, \hat{q}^*_{p,B}$ as the LCL and UCL, respectively. The method to find sample quantiles proposed by Hyndman and Fan (1996) will be used for the simulation study in Section 4.

The above bootstrap chart is called MLE-b chart. Similarly, if the MLEs, $\hat{\alpha}(\hat{\alpha}^*)$ and $\hat{\lambda}(\hat{\lambda}^*)$, of α and λ are replaced by the MMEs, $\tilde{\alpha}(\tilde{\alpha}^*)$ and $\tilde{\lambda}(\tilde{\lambda}^*)$, respectively, and MLE method is replaced by MME method from Step 1 to Step 3, then the corresponding bootstrap chart is constructed based on moment method, and is called MME-b chart. The plot statistic for MLE-b chart is $\hat{Q}_{p,m}(\hat{\Theta}_m)$ and the plot statistic for MME-b chart is $\tilde{Q}_{p,m}(\tilde{\Theta}_m)$.

4. Simulation study

To examine the performance of three BTXII percentile control charts discussed in Section 3, an intensive Monte Carlo simulation study was conducted using R language which was originally developed by Ihaka and Gentleman (1996). The R source codes can be obtained from authors upon request.

The performance of BTXII percentile control charts are investigated in terms of simulated ARL_0 and ARL_1 and the standard errors of run lengths (SERLs), respectively. Moreover, the average of upper control limits (UCLs), the average of lower control limits (LCLs), and their associated standard errors are also evaluated through the simulation. Simulation has been carried out with different sample sizes (specially, sample sizes 4, 5 and 6 are considered), different percentiles of interest, and different levels of FARs. Five thousand bootstrap repetitions, $B = 5000$, have been used to determine the control limits for each bootstrap chart. Moreover, all complete procedures described in Section 3 for each control chart have been repeated five thousand times to evaluate the ARL value, the associated SERL value and the standard errors of control limits. For brevity, some simulation results are displayed in Table 1 to Table 12.

Tables 1 and 2 show that the simulated ARL_0 and the corresponding SERL for the Shewhart-type control charts. These two tables indicate that Shewhart-type control chart seriously underestimate the nominal ARL_0 due to narrow band of control limits that produced in general. This means that the Shewhart-type chart will incur a higher FAR than the expectation. Tables 3, 5, 6, 8, 9, 10 and 11 show that the simulated ARL_0 and the corresponding SERL for MLE-b and MME-b charts. Generally, MLE-b and MME-b charts have ARL_0 closer to their corresponding nominal ARL_0s than the Shewhart-type control charts for the same BTXII distribution. Therefore, it is clear that MLE-b and MME-b charts outperform the Shewhart-type chart in terms of the simulated ARL_0. Although the MLE-b chart and MME-b chart perform satisfactory in terms of the simulated ARL_0 when they are compared with the Shewhart-type chart, there are some differences between the MLE-b chart and the MME-b chart. Tables 6, 8 and 10 show that MLE-b charts have simulated ARLs generally much higher than the corresponding nominal ARL_0s for FAR=0.002 or FAR = 0.0027 when $\alpha < 1$, and Table 3 shows that MLE-b charts have simulated ARLs generally smaller than the corresponding nominal ARL_0 when $\alpha > 1$. Table 5 shows that the MME-b chart has simulated ARLs generally much larger than the corresponding nominal ARL_0 for FAR=0.0027 when $\alpha > 1.0$; and Table 9 shows that the MME-b chart has simulated ARLs generally much smaller than the corresponding nominal ARL_0s for FAR=0.0027 and FAR=0.002 when $\alpha < 1.0$. Tables 4 and 7 show the average values of simulated LCLs, simulated UCLs, and their associated standard errors (SDER) when the MLE-b chart is used. It can be seen that the standard errors of the LCL and UCL are generally smaller for MLE-b chart. It can be easily to check that the coefficient of variation, SDER/average value, is smaller than 0.009 for almost all of cases except those cases with $p = 0.01$. For all those cases with $p = 0.01$, the coefficient of variation is still below 0.05. That is, the proposed constructing procedures for MLE-b charts can provide stable control limits and give a helpful guidance to construct MLE-b charts in the practical applications. Since the UCL of MME-b chart when γ_0=0.0027 and γ_0=0.002 can be infinite, the MME-b chart will not be suggested to be used for monitoring the BTXII percentile in the practice. However, because the MLE-b and MME-b charts both have ARL_0s closer to the corresponding nominal ART_0s than the Shewhart-type chart, MLE-b and MME-b charts will be proposed for the further investigation to exam the ARL for out-of-control process.

The main concern is the downward shift of distribution percentile which indicates a deteriorating quality in the product lifetime. First, control limits of the MLE-b chart and MME-b chart are established based on generated in-control Phase I subgroups. Then further subgroups are generated from an out-of-control process and used for evaluating the ARL_1 and its standard error. As mentioned before that for a given $0 < p < 1$, the BTXII percentile, $Q(p;\alpha,\lambda)$, is a decreasing function of α when λ is treated as a fixed positive number, and is an increasing function of λ when the given value of α is greater than $-ln(1-p)/ln(2)$, else is an decreasing function of λ when the given value of α is smaller than $-ln(1-p)/ln(2)$. Hence, there are at least three possible ways to implement an out-of-control process from the in-control process. We may let the value of λ_0 of the in-control process fixed and simply increase the value of α, from α_0 value for the in-control process to a larger value α_1 for an out-of-control process. If the value of α_0 of the in-control process is fixed, then we can simply decrease the value of λ, from λ_0 for the in-control process to a smaller value of λ_1 for an out-of-control if $\alpha_0 > -ln(1-p)/ln(2)$; or increase the value of λ, from λ_0 for the in-control process to a larger value of λ_1 for an out-of-control if $\alpha_0 < -ln(1-p)/ln(2)$. It should be mentioned that it is difficult to signal an out-of-control that is caused by increasing α alone because the corresponding ARL_1 is usually very large based on the simulation experience. Hence, α will be treated as the fixed constant from in-control process to examine the performance of monitoring out-of-control.

For brevity, part of simulation results for monitoring out-of-control cases with α_0 fixed and is greater than $-ln(1-p)/ln(2)$ are displayed in Table 12. In view of Table 12, the ARL_1 values and the associated SERLs are very small. Therefore, these simulation results support the fact that both MLE-b chart and MME-b chart are capable of monitoring the downward shift of BTXII percentiles due to the downward shift of parameter λ. Table 12 also shows that ARL_1 decreases as λ_1 further decreases from in-control process with a value of λ_0.

5. Illustrative examples

Wingo (1993) assessed the reliability of a certain electronic component by using BurrXII distribution. Based on the twenty failure times in terms of months from a total of thirty electronic components in the life test, Wingo (1993) indicated that the BurrXII distribution was a good lifetime model and obtained the maximum likelihood estimates of λ and α to be 1.29 and 0.64, respectively. Soliman (2002) also investigated BurrXII reliability estimation through maximum likelihood and Bayesian approaches based on the same data set.

In this section, the MLE-b chart is applied to monitoring the lifetime quality of the electronic component used by Wingo (1993). It is also assumed that the quality engineer can use accelerate life testing process to project the true lifetime of the test items during the quality monitoring process. Since the original lifetime data sets of electronic components mentioned above was not originally for the purpose of constructing control charts, the data set cannot be used directly for the quality control study.

To implement the process of monitoring the quality of the electronic component lifetime, twenty subgroups of each six electronic component lifetimes are simulated independently from an in-control BurrXII process with $\alpha_0 = 0.64$ and $\lambda_0 = 1.29$, of which tenth percentile is found to be $Q(0.10, \alpha, \lambda) = 0.263$. These twenty in-control subgroups of each six lifetimes are reported in Table 13. Assuming that the process parameter λ shifts to $\lambda_1 = 0.65$ after the first twenty in-control subgroups, another twenty out-of-control subgroups of each six electronic lifetimes are generated from the BurrXII distribution with $\alpha = 0.64$ and $\lambda = 0.65$ and reported in Table 14. The MLE-b chart is established based on the twenty in-control subgroups of each six lifetimes that are reported in Table 13, with FAR=0.0027 and $B = 5000$. The control limits of the MLE-b chart are obtained as

$$\mathrm{UCL}_{\mathrm{MLE-b}} = 1.359,$$
$$\mathrm{LCL}_{\mathrm{MLE-b}} = 0.0201,$$

and the CL of the MLE-b chart is $\mathrm{CL}_{\mathrm{MLE}} = 0.33$. Figure 1 shows that the MLE-b chart provides asymmetric control limits from the CL and the first out-of-control signal is observed immediately right after the process shifted. Hence, it is clear that this MLE-b chart can efficiently indicate the process out-of-control.

6. Conclusions

To monitor the BTXII percentiles, a Shewhart-type chart and two PBCs have been constructed. The Shewhart-type control chart is constructed based on the asymptotic normal distribution of maximum likelihood estimator and delta method. Because the Shewhart-type chart cannot provide adequate control limits that have been shown by the simulated average running length, also the upper limit of MME-b chart is not stable and can be infinite when FAR is small, PBCs based on MLE is the only control chart to be proposed for monitoring BTXII percentiles. Through an intensive Monte Carlo simulation, it has been found that the MLE-b chart is easy to be constructed for subgroup of size small (such as 3, 4, 5 or 6) and the MLE-b chart can efficiently signal out-of-control when the process shifts to out-of-control. Therefore, the MLE-b chart would be recommended for monitoring BTXII percentiles in the practice.

Extending the developed procedures of control charts in Section 3 for monitoring the percentiles of other important life distributions is of great interest and will be investigated in the future.

Table 1. Shewhart in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 5.49$, $\lambda = 0.85$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	1.3834	0.01191	1.7992	0.02177	2.3878	0.03124
$p = 0.05$	4.7034	0.05913	6.4733	0.08110	7.889	0.09296
$p = 0.10$	8.9379	0.10820	10.5607	0.12142	12.190	0.14132
$p = 0.25$	19.6438	0.23898	23.3004	0.2848	26.363	0.34117
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	2.6369	0.03997	4.0760	0.05599	5.8900	0.07793
$p = 0.05$	9.8878	0.12098	12.5429	0.15222	14.862	0.18475
$p = 0.10$	17.185	0.22091	21.0118	0.26586	25.5531	0.34156
$p = 0.25$	55.495	0.78871	74.085	1.0734	91.3495	1.2963
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	3.4933	0.05536	5.322	0.07106	7.4185	0.0944
$p = 0.05$	11.942	0.15227	15.164	0.18537	18.543	0.2368
$p = 0.10$	22.235	0.30180	28.048	0.37260	35.492	0.4938
$p = 0.25$	86.445	1.21478	122.557	1.8980	159.891	2.3877
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	3.6757	0.05812	5.5902	0.07377	7.6828	0.09680
$p = 0.05$	12.444	0.15914	15.856	0.19850	19.463	0.25396
$p = 0.10$	23.422	0.31607	29.751	0.39609	38.131	0.5327
$p = 0.25$	95.806	1.38385	135.526	2.2634	179.328	2.745

Table 2. Shewhart in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 0.6287$, $\lambda = 1.1953$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	1.1138	0.00508	1.1708	0.00725	1.2406	0.00855
$p = 0.05$	1.3722	0.01051	1.4706	0.01244	1.5684	0.01391
$p = 0.10$	1.5316	0.01301	1.5922	0.01389	1.6970	0.01574
$p = 0.25$	3.9232	0.13032	3.5044	0.05024	3.3582	0.04358
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	1.2542	0.00942	1.4296	0.01567	1.7204	0.02388
$p = 0.05$	2.0212	0.02372	2.3784	0.03013	2.7950	0.03666
$p = 0.10$	2.4292	0.02735	2.7394	0.03325	3.0014	0.03788
$p = 0.25$	25.8350	0.98667	22.5360	0.90012	18.7290	0.57255
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	1.3822	0.01475	1.7506	0.02571	2.1444	0.03399
$p = 0.05$	2.6836	0.03549	3.2262	0.04441	3.9374	0.05405
$p = 0.10$	3.3146	0.04089	3.6928	0.04667	4.2558	0.05644
$p = 0.25$	45.9794	2.24265	39.5150	1.49021	35.0228	1.29941
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	1.4190	0.01563	1.8192	0.02711	2.2538	0.03617
$p = 0.05$	2.8682	0.03879	3.4478	0.04721	4.2344	0.05824
$p = 0.10$	3.5734	0.04458	3.9568	0.05029	4.5810	0.06187
$p = 0.25$	52.1056	2.52317	45.3782	1.96845	38.9896	1.43637

Table 3. MLE in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 5.49$, $\lambda = 0.85$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	9.489	0.0919	9.302	0.0892	9.280	0.0894
$p = 0.05$	9.451	0.0919	9.494	0.0922	9.253	0.0905
$p = 0.10$	9.382	0.0940	9.389	0.0930	9.151	0.0898
$p = 0.25$	9.551	0.0955	9.284	0.0915	9.374	0.0937
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	91.670	1.0159	91.756	1.0008	89.876	0.9435
$p = 0.05$	92.312	1.0736	92.202	1.0292	91.252	1.0344
$p = 0.10$	92.351	1.0957	93.422	1.0592	92.104	1.0469
$p = 0.25$	96.018	1.2147	92.156	1.0852	90.236	1.0322
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	350.522	4.3742	339.584	4.0075	335.258	3.8509
$p = 0.05$	352.948	4.6279	347.647	4.3994	342.048	4.2707
$p = 0.10$	360.585	5.0646	355.758	4.6781	340.129	4.209
$p = 0.25$	384.143	5.8859	351.579	4.5827	348.172	4.6368
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	473.976	6.0846	463.632	5.5502	455.862	5.4084
$p = 0.05$	487.756	7.1034	474.699	6.1434	459.590	5.9753
$p = 0.10$	493.142	7.0268	494.083	6.6919	456.733	5.697
$p = 0.25$	517.556	8.2068	485.581	6.7314	478.182	6.737

Table 4. MLE in-control LCL and UCL estimates and (SDER) for Burr XII ($\alpha = 5.49$, $\lambda = 0.85$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	0.000034	0.03642	0.00004	0.02390	0.000048	0.01742
	(0.0000003)	(0.000079)	(0.0000003)	(0.000053)	(0.0000003)	(0.000038)
$p = 0.05$	0.000470	0.07073	0.00054	0.05284	0.000620	0.04260
	(0.0000027)	(0.000126)	(0.0000027)	(0.000093)	(0.0000027)	(0.000073)
$p = 0.10$	0.001480	0.09719	0.00170	0.07710	0.001917	0.06462
	(0.0000070)	(0.000161)	(0.0000068)	(0.000121)	(0.0000070)	(0.000098)
$p = 0.25$	0.007184	0.16155	0.00818	0.13717	0.009054	0.12151
	(0.0000238)	(0.000238)	(0.0000241)	(0.000187)	(0.0000234)	(0.000156)
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	0.000001	0.10625	0.000003	0.0698	0.000004	0.05015
	(0.00000002)	(0.000182)	(0.00000003)	(0.000119)	(0.00000004)	(0.000085)
$p = 0.05$	0.000049	0.16331	0.000076	0.1193	0.000109	0.09356
	(0.0000004)	(0.000255)	(0.00000052)	(0.000177)	(0.0000006)	(0.000134)
$p = 0.10$	0.000227	0.20340	0.000336	0.1562	0.000455	0.12714
	(0.0000015)	(0.000313)	(0.0000018)	(0.000219)	(0.000002)	(0.000168)
$p = 0.25$	0.001914	0.29678	0.002596	0.2423	0.003252	0.20822
	(0.0000085)	(0.000446)	(0.0000099)	(0.000321)	(0.000011)	(0.000257)
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	0.0000003	0.1597	0.0000007	0.10536	0.000001	0.0757
	(0.00000001)	(0.000279)	(0.000000009)	(0.000175)	(0.00000001)	(0.000125)
$p = 0.05$	0.0000150	0.2283	0.0000278	0.16530	0.000045	0.1286
	(0.00000015)	(0.000378)	(0.00000023)	(0.000249)	(0.0000003)	(0.000185)
$p = 0.10$	0.0000858	0.2752	0.0001458	0.20869	0.000218	0.1681
	(0.0000007)	(0.000445)	(0.00000091)	(0.000304)	(0.0000012)	(0.000229)
$p = 0.25$	0.0009619	0.3831	0.0014421	0.30796	0.001932	0.2617
	(0.000005)	(0.00062)	(0.0000064)	(0.000432)	(0.0000073)	(0.000338)
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	0.0000002	0.1735	0.0000005	0.11456	0.0000010	0.0823
	(0.000000003)	(0.00031)	(0.000000007)	(0.000194)	(0.00000001)	(0.000137)
$p = 0.05$	0.0000115	0.2447	0.0000223	0.17678	0.0000367	0.1372
	(0.00000012)	(0.000415)	(0.00000019)	(0.000270)	(0.00000026)	(0.000200)
$p = 0.10$	0.0000692	0.2931	0.0001213	0.22163	0.0001855	0.1780
	(0.0000006)	(0.000486)	(0.00000079)	(0.000330)	(0.000001)	(0.000245)
$p = 0.25$	0.0008256	0.4039	0.0012658	0.32371	0.0017193	0.2744
	(0.0000045)	(0.000666)	(0.0000058)	(0.000463)	(0.0000067)	(0.000360)

Table 5. MME in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 5.49$, $\lambda = 0.85$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	9.7176	0.12853	9.7766	0.13270	9.5194	0.13270
$p = 0.05$	9.6360	0.13179	9.7266	0.12734	9.4440	0.12713
$p = 0.10$	9.6993	0.13286	9.7870	0.13396	9.7010	0.13176
$p = 0.25$	9.5878	0.13294	10.1604	0.14027	9.5528	0.12966
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	99.668	1.49705	97.1502	1.49046	90.5162	1.32451
$p = 0.05$	96.1738	1.49783	97.4186	1.53433	92.3276	1.4170
$p = 0.10$	99.0564	1.60879	99.7884	1.60474	95.7676	1.48037
$p = 0.25$	101.8258	1.73413	102.8149	1.71767	98.4534	1.64179
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	368.3414	6.15738	360.3882	6.14465	354.4038	5.80779
$p = 0.05$	384.0682	6.72020	380.5550	6.75787	353.4776	6.15968
$p = 0.10$	387.70900	7.26753	383.1900	6.93256	369.1508	6.43089
$p = 0.25$	416.5502	8.39249	410.6444	8.35209	393.9396	7.59261
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	513.3276	9.08390	492.6596	8.89021	477.9740	7.91654
$p = 0.05$	530.7884	10.13229	522.7432	9.68210	499.9750	9.07411
$p = 0.10$	539.8073	10.47616	520.9810	9.60464	513.6212	9.62915
$p = 0.25$	584.4068	14.19618	547.3264	10.60332	545.0452	11.28614

Table 6. MLE in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 0.08$, $\lambda = 5.47$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	9.5298	0.1412	9.3873	0.1384	9.6031	0.1420
$p = 0.05$	9.6710	0.1541	9.6918	0.1584	9.6277	0.1509
$p = 0.10$	9.7906	0.1546	9.5872	0.1527	9.5037	0.1516
$p = 0.25$	9.9644	0.1475	9.5256	0.1345	9.6662	0.1363
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	96.0906	2.0424	96.1712	2.0356	97.3494	2.1109
$p = 0.05$	95.7430	2.2902	100.1690	2.3573	102.6900	2.6672
$p = 0.10$	103.1736	2.7254	100.1868	2.5098	99.7786	2.4224
$p = 0.25$	101.1328	2.2240	99.5562	2.2487	101.1302	1.9161
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	377.6082	10.4431	375.0289	9.8843	381.3017	10.3995
$p = 0.05$	372.4666	11.4847	390.7094	12.1085	429.0386	23.6215
$p = 0.10$	439.1992	23.7574	409.2720	14.3653	384.6976	11.0866
$p = 0.25$	401.9918	13.7038	382.2936	10.7416	389.5548	10.3047
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	523.8520	15.1520	534.2442	16.9458	532.6824	16.0324
$p = 0.05$	523.2488	19.0018	558.5619	19.5558	584.0950	26.5399
$p = 0.10$	617.2196	31.7013	568.7408	19.7148	540.4306	17.4381
$p = 0.25$	566.3458	21.5265	526.1670	15.8597	530.0650	14.8798

Table 7. MLE in-control LCL and UCL estimates and (SDER) for Burr XII ($\alpha = 0.08$, $\lambda = 5.47$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	0.331147	1.016391	0.322579	1.017691	0.322958	1.018548
	(0.002597)	(0.000063)	(0.002083)	(0.000045)	(0.001774)	(0.000034)
$p = 0.05$	0.589821	1.225620	0.598627	1.214041	0.610306	1.205447
	(0.002422)	(0.000466)	(0.002073)	(0.000371)	(0.001809)	(0.000317)
$p = 0.10$	0.784760	1.573339	0.809836	1.532531	0.831728	1.504863
	(0.002064)	(0.001179)	(0.001819)	(0.000967)	(0.001614)	(0.000815)
$p = 0.25$	1.141261	3.583147	1.188931	3.308213	1.232259	3.146491
	(0.001004)	(0.007366)	(0.000885)	(0.005624)	(0.000796)	(0.004742)
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	0.120004	1.036936	0.127383	1.035598	0.137907	1.034597
	(0.000170)	(0.000087)	(0.001342)	(0.000065)	(0.001150)	(0.000052)
$p = 0.05$	0.317830	1.350468	0.344782	1.318678	0.369802	1.296670
	(0.002155)	(0.000730)	(0.001870)	(0.000565)	(0.001659)	(0.000469)
$p = 0.10$	0.502754	1.917649	0.545498	1.814336	0.581290	1.746971
	(0.002251)	(0.00208)	(0.002019)	(0.001598)	(0.001807)	(0.001289)
$p = 0.25$	0.878338	6.162181	0.931290	5.243810	0.971345	4.719448
	(0.001514)	(0.018527)	(0.001143)	(0.012702)	(0.000831)	(0.009859)
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	0.074541	1.047298	0.081486	1.044338	0.091395	1.042335
	(0.001338)	(0.000107)	(0.001060)	(0.000079)	(0.000914)	(0.000067)
$p = 0.05$	0.236981	1.422078	0.264146	1.376609	0.289620	1.346390
	(0.002002)	(0.000924)	(0.001739)	(0.000688)	(0.001550)	(0.000568)
$p = 0.10$	0.405450	2.131727	0.450221	1.981210	0.487617	1.887025
	(0.002256)	(0.002766)	(0.002024)	(0.002033)	(0.001817)	(0.001612)
$p = 0.25$	0.776175	8.264172	0.839282	6.687169	0.890617	5.840448
	(0.001846)	(0.030349)	(0.001514)	(0.019440)	(0.001187)	(0.014458)
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	0.067201	1.049665	0.073955	1.046294	0.083481	1.044049
	(0.001264)	(0.000132)	(0.001004)	(0.000083)	(0.000869)	(0.000067)
$p = 0.05$	0.222260	1.438751	0.248986	1.389969	0.274334	1.357823
	(0.001965)	(0.000977)	(0.001709)	(0.000719)	(0.001525)	(0.000594)
$p = 0.10$	0.386727	2.184468	0.431582	2.020981	0.469200	1.919738
	(0.002253)	(0.002962)	(0.002021)	(0.002154)	(0.001814)	(0.001699)
$p = 0.25$	0.754931	8.848224	0.818745	7.071897	0.872401	6.138254
	(0.001900)	(0.034122)	(0.001585)	(0.021456)	(0.001266)	(0.015947)

Table 8. MLE in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 0.6287$, $\lambda = 1.1953$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	9.4150	0.13201	9.6220	0.13632	9.3410	0.12929
$p = 0.05$	9.5638	0.14381	9.6684	0.13894	9.4558	0.13258
$p = 0.10$	9.4874	0.14721	9.6500	0.14665	9.4704	0.13930
$p = 0.25$	9.6636	0.15440	9.7216	0.14783	9.5822	0.14184
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	100.7326	1.82686	100.7298	1.77273	98.3096	1.66977
$p = 0.05$	107.7196	2.34663	105.9360	2.15765	103.5364	2.09587
$p = 0.10$	102.2632	2.26380	103.5140	2.14112	103.3632	2.10837
$p = 0.25$	104.0292	2.25916	102.1952	2.26906	100.1510	1.94429
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	397.9636	9.07985	404.1122	8.95056	389.0514	7.94027
$p = 0.05$	454.1206	12.97278	436.5634	10.64892	426.0436	10.53851
$p = 0.10$	416.3770	12.23871	421.1686	10.99402	413.1744	10.28794
$p = 0.25$	424.0622	11.89269	426.5126	10.55059	418.8490	10.92879
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	550.3726	12.93418	566.3390	13.19474	547.2370	12.24395
$p = 0.05$	629.9942	18.61569	613.0124	17.66224	609.3994	16.24447
$p = 0.10$	583.6988	17.21763	583.3014	18.56409	578.4706	15.76502
$p = 0.25$	604.0076	17.70668	599.6066	17.75492	573.1262	14.75778

Table 9. MME in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 0.6287$, $\lambda = 1.1953$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	10.2918	0.15131	10.1830	0.15006	10.0550	0.13937
$p = 0.05$	10.1550	0.14767	10.1682	0.15174	9.9608	0.13781
$p = 0.10$	10.4208	0.15280	10.2852	0.14984	9.8646	0.14093
$p = 0.25$	10.2948	0.15396	10.1638	0.15112	10.1486	0.15198
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	127.3104	2.61127	122.3928	2.46522	120.3358	2.4614
$p = 0.05$	120.2924	2.34216	115.9948	2.25263	113.2856	2.33937
$p = 0.10$	122.2716	2.41879	119.6858	2.43758	114.2008	2.21692
$p = 0.25$	112.4390	2.12890	114.2208	2.22453	107.8744	2.03964
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	310.4800	5.35882	318.1268	5.35569	350.3406	6.16744
$p = 0.05$	303.2566	5.22455	316.4058	5.38817	331.1858	5.80927
$p = 0.10$	306.0644	5.15079	315.5124	5.45777	335.1776	5.95371
$p = 0.25$	290.5098	5.00318	306.2186	5.45141	326.5658	5.84490
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	350.7946	5.86756	363.8666	5.88299	412.4976	6.98893
$p = 0.05$	341.6172	5.68974	362.2930	5.96056	389.8286	6.56699
$p = 0.10$	348.7050	5.66114	361.2808	6.05692	395.7262	6.87873
$p = 0.25$	330.2448	5.41132	349.1800	5.90239	383.8220	6.66151

Table 10. MLE in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 0.64$, $\lambda = 1.29$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	9.561	0.1340	9.451	0.1333	9.3498	0.1302
$p = 0.05$	9.721	0.1476	9.716	0.1457	9.5574	0.1392
$p = 0.10$	9.707	0.1523	9.582	0.1430	9.5218	0.1353
$p = 0.25$	9.736	0.1512	9.673	0.1453	9.7502	0.1464
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	99.992	1.4071	97.472	1.6857	96.4556	1.6311
$p = 0.05$	111.907	2.8284	104.834	2.2176	106.851	2.3239
$p = 0.10$	106.881	2.4504	99.306	2.0311	102.798	2.0213
$p = 0.25$	102.712	2.2068	103.154	2.0665	102.795	2.1567
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	418.510	13.599	409.033	9.259	393.928	8.7907
$p = 0.05$	434.100	12.515	436.393	10.806	451.500	32.928
$p = 0.10$	426.500	12.224	418.756	11.107	420.345	9.972
$p = 0.25$	440.226	12.885	431.731	11.547	410.829	10.720
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	567.939	17.3066	575.799	13.4838	557.086	13.4318
$p = 0.05$	620.678	19.4844	598.680	15.4241	616.803	34.827
$p = 0.10$	607.643	18.8936	576.826	16.2036	588.089	14.549
$p = 0.25$	619.974	18.596	601.064	17.0387	573.126	15.860

Table 11. MME in-control ARL estimate and its corresponding SD for Burr XII ($\alpha = 0.64$, $\lambda = 1.29$) percentiles and $\gamma_0 = 0.1, 0.01, 0.0027, 0.002$ FAR's (Twenty subgroups, $k = 20$).

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SERL	ARL	SERL	ARL	SERL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	10.547	0.154	10.066	0.148	9.792	0.142
$p = 0.05$	10.317	0.149	10.087	0.146	9.963	0.144
$p = 0.10$	10.392	0.151	10.005	0.144	10.079	0.147
$p = 0.25$	10.094	0.154	10.329	0.1556	9.984	0.146
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	126.390	2.882	121.289	2.453	118.649	2.289
$p = 0.05$	127.755	2.536	124.592	2.509	115.1036	2.240
$p = 0.10$	130.878	2.862	119.787	2.660	117.393	2.269
$p = 0.25$	120.317	2.734	114.632	2.271	113.353	2.332
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	417.422	8.092	412.251	8.076	410.480	7.964
$p = 0.05$	438.158	8.979	420.947	8.259	408.486	8.191
$p = 0.10$	411.924	7.983	409.484	8.352	416.038	8.445
$p = 0.25$	392.707	7.666	398.708	7.924	394.587	7.707
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	510.375	9.516	501.348	9.363	522.244	9.582
$p = 0.05$	522.896	10.286	503.637	9.393	515.792	9.974
$p = 0.10$	496.208	9.040	498.247	9.599	527.908	10.212
$p = 0.25$	472.369	8.811	489.305	9.155	496.125	9.383

Table 12. Simulated ARL_1 and SERL values when λ shifts to λ_1 from $\lambda_0 = 1.29$ for $m = 6$, $k = 20$ and $\alpha = 0.64$.

p	γ	λ_0	λ_1	ARL_1(MLE)	SERL	ARL_1(MME)	SERL
0.10	0.10	1.29	0.65	2.1824	0.0237	2.1454	0.0233
		1.29	0.43	1.4780	0.0121	1.4036	0.0109
		1.29	0.32	1.2898	0.0088	1.2318	0.0077
		1.29	0.26	1.2142	0.0069	1.1536	0.0059
		1.29	0.21	1.1478	0.0057	1.1076	0.0048
0.10	0.01	1.29	0.65	4.3208	0.0237	4.3286	0.0608
		1.29	0.43	1.9874	0.0203	1.8922	0.0190
		1.29	0.32	1.5380	0.0133	1.4714	0.0120
		1.29	0.26	1.3774	0.0105	1.2940	0.0087
		1.29	0.21	1.2620	0.0081	1.1962	0.0068
0.10	0.0027	1.29	0.65	6.4236	0.0908	6.8638	0.1068
		1.29	0.43	2.4112	0.0272	2.3438	0.0269
		1.29	0.32	1.7168	0.0163	1.6520	0.0152
		1.29	0.26	1.4925	0.0125	1.3860	0.0105
		1.29	0.21	1.3314	0.0095	1.2596	0.0080
0.10	0.002	1.29	0.65	7.1050	0.1043	7.6462	0.1226
		1.29	0.43	2.5344	0.0290	2.4636	0.0286
		1.29	0.32	1.7614	0.0171	1.6956	0.0157
		1.29	0.26	1.5205	0.0129	1.4122	0.0108
		1.29	0.21	1.3506	0.0097	1.2784	0.0084
0.25	0.1	1.29	0.65	2.3520	0.0251	2.4080	0.0261
		1.29	0.43	1.6628	0.0151	1.6188	0.0140
		1.29	0.32	1.4186	0.0111	1.4112	0.0111
		1.29	0.26	1.3160	0.0092	1.2960	0.0088
		1.29	0.21	1.2496	0.0077	1.2208	0.0077
0.25	0.01	1.29	0.65	4.8886	0.0634	5.1186	0.0681
		1.29	0.43	2.3712	0.0256	2.3986	0.0271
		1.29	0.32	1.8024	0.0175	1.8144	0.0178
		1.29	0.26	1.5992	0.0140	1.5524	0.0134
		1.29	0.21	1.4554	0.0113	1.3958	0.0106
0.25	0.0027	1.29	0.65	7.4906	0.1049	8.4132	0.1302
		1.29	0.43	2.9454	0.0337	3.1672	0.0393
		1.29	0.32	2.0610	0.0217	2.1784	0.0243
		1.29	0.26	1.7802	0.0168	1.7728	0.0169
		1.29	0.21	1.5694	0.0132	1.5776	0.0138
0.25	0.002	1.29	0.65	8.4338	0.1210	9.6480	0.1563
		1.29	0.43	3.1014	0.0365	3.3952	0.0425
		1.29	0.32	2.1274	0.0226	2.3306	0.0273
		1.29	0.26	1.8194	0.0174	1.8750	0.0189
		1.29	0.21	1.5958	0.0136	1.6528	0.0148

Table 13. Top twenty subgroups of electronic component lifetimes generated from the BTXII distribution with $\alpha_0 = 0.64$ and $\lambda = 1.29$.

Subgroup number	lifetime observations					
1	0.545523	0.111869	3.562735	4.468829	0.481474	1.897821
2	1.021759	17.51915	0.272386	0.881968	5.378379	0.382019
3	9.267913	31.51052	2.834265	15.74311	1.191486	0.205128
4	2.335865	19.62976	0.729407	3.788054	3.209420	0.514025
5	0.920839	3.458564	0.095770	1.394487	5.919687	0.201121
6	1.197991	8.852825	2.452986	8.003429	0.444778	3.099667
7	1.668747	0.483094	2.756992	4.265422	0.969981	0.779199
8	1.365622	2.653202	7.651898	3.170058	2.745687	3.548740
9	1.256711	0.743504	0.768741	0.218993	0.857355	3.861241
10	4.032995	2.281709	8.527115	0.132912	3.334109	138.6888
11	2.469075	1.610966	20.34522	0.264265	6.221741	65.97923
12	2.925442	1.021302	11.68723	0.782228	2.034763	5.303781
13	6.298546	26.79470	7.331502	4.307616	0.280723	0.839558
14	0.142750	3.693631	17.18273	4.659451	10.20538	1.745063
15	17.08135	30.98131	1.045303	5.300879	28.17504	2.037199
16	38.20007	0.736304	0.637200	109.0852	0.409012	0.082647
17	1.785511	1.043847	0.346231	0.673403	350.9389	5.438014
18	1.882098	2.347502	0.297302	4.046231	1.655644	31.81990
19	3.055799	13.70062	0.573477	0.211356	1.449541	1.484838
20	0.475345	0.414347	1.672673	4.536216	0.802389	0.489192

Table 14. Twenty out-of-control subgroups of electronic component lifetimes generated from the BTXII distribution with $\alpha_1 = 0.64$ and $\lambda = 0.65$.

Subgroup number	lifetime observations					
21	0.011139	0.603066	1790.836	4.850877	0.124040	0.172081
22	76.13965	15.24739	1.163399	1.017178	116.4952	195.5026
23	3038.643	93.54499	9.263096	2.691029	10.17255	1.975991
24	5130.794	47.76424	0.071867	1.241000	1.666116	1.180654
25	0.262417	0.009454	0.772823	4.227578	579.6631	3.003024
26	0.064320	7057.562	85.67451	42.93172	32.71423	1.157400
27	6.807197	0.186522	5.003434	0.622301	2039.907	0.969570
28	2.346855	35.77861	303.1503	0.795887	0.203652	7.285395
29	5.439290	2.678413	247.7381	39.21904	0.305026	1.086270
30	0.276058	29.53337	0.844077	1.415701	7.046816	38764.32
31	66.64765	0.099803	0.252450	4.347114	5.626255	0.429788
32	0.314641	8.444252	0.316886	330.7342	9.144874	1.049583
33	0.966376	7.771045	121.9134	0.062052	3.787511	5131.382
34	0.225502	3.796028	0.019064	0.042123	0.162977	0.775827
35	0.182925	0.000217	6392.907	264.1949	862.0526	1.529213
36	0.521124	0.007930	15.57398	7.756012	5.296139	4686.697
37	0.690275	0.008445	0.024687	0.057587	86.76206	0.238617
38	0.183715	2045.861	13.17467	899.0267	7.692057	6.137891
39	0.722723	44.07850	52.17710	2.564983	49.58774	4192.347
40	2.598001	220.7679	5.439253	14.08382	18.34224	1.137503

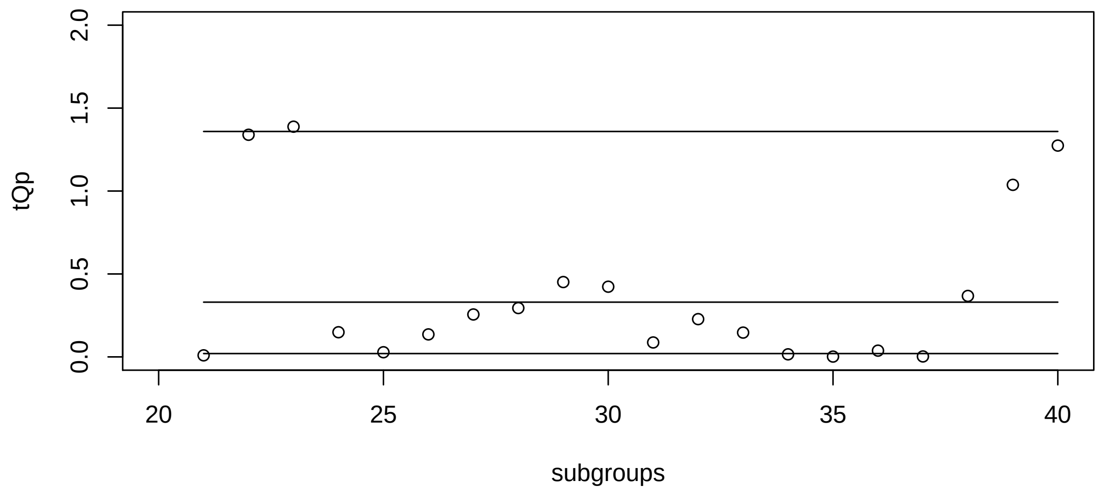

Figure 1. The MLE-b chart for the lifetime data of electronic component with FAR=0.0027.

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References

AL-Hussaini, E.K., Ali Mousa, M.A.M., 1992. Estimation under the Burr type XII failure model based on censored data: a comparative study. Test, 1, 47-60.

Ali Mousa M.A.M., Jaheen Z.F., 2002. Statistical inference for the Burr model based on progressively censored data. Computers & Mathematics with Applications, 43, 1441-1449.

Bajgier, S. M., 1992. The use of bootstrapping to construct limits on control charts. Proceedings of the Decision Science Institute, San Diego,CA, 1611-1613.

Burr, I. W., 1942. Cumulative frequency functions. Annals of Mathematical Statistics, 13, 215-222.

Chen, F.-L.,Yeh, C.H., 2006. Economic design of control charts with Burr distribution for non-normal data under Weibull failure mechanism. Journal of the Chinese Institute of Industrial Engineers, 23, 200-206.

Efron, B.,Tibshirani, R. J., 1993. An Introduction to the Bootstrap. Chapman and Hall, New York.

Gunter, B., 1992. Bootstrapping: How to make something from almost nothing and get statistically valid answers, Part III. Quality Progress, 25, 119-122.

Hyndman, R. J., Fan, Y., 1996. Sample quantiles in statistical packages. American Statistician, 50, 361-365.

Ihaka, R., Gentleman, R., 1996. R: A language for data analysis and graphics. Journal of Computational and Graphical Statistics, 5, 299-314.

Jones, L. A.,Woodall, W. Y., 1998. The performance of bootstrap control charts. Journal of Quality Technology, 30, 362-375.

Lio, Y. L.,Park, C., 2008. A bootstrap control chart for Birnbaum-Saunders percentiles. Quality and Reliability Engineering International, 24, 585-600.

Lio, Y. L.,Park, C., 2010. A bootstrap control chart for inverse Gaussian percentiles. Journal of Statistical Computation and Simulation, 80, 287-299.

Lio, Y.L.,Tsai, T.-R., 2012. Estimation of $\delta = P(X < Y)$ for Burr XII distribution based on the progressively first failure-censored samples. Journal of Applied Statistics, 39, 309-322.

Lio, Y.L.,Tsai, T.-R.,Aslam, M.,Jiang, N., 2014. Control Charts for Monitoring Burr Type-X Percentiles. Communications in Statistics-Simulation and Computation, 43, 761-776.

Lio, Y.L.,Tsai, T.-R., Wu, S.-J., 2010. Acceptance sampling plans from truncated life tests based on the Burr type XII percentiles. Journal of the Chinese Institute of Industrial Engineers, 27, 270-280.

Liu, R. Y.,Tang, J., 1996. Control charts for dependent and independent measurements based on the bootstrap. Journal of the American Statistical Association, 91, 1694-1700.

Moore, D.,Papadopoulos, A.S., 2000. The Burr type XII distribution as a failure model under various loss functions. Microelectronics and Reliability, 40, 2117-2122.

Nichols, M. D.,Padgett, W. J., 2005. A bootstrap control chart for Weibull percentiles. Quality and Reliability Engineering International, 22, 141-151.

Onar, A.,Padgett, W. J., 2000. Accelerated test models with the inverse Gaussian distribution. Journal of Statistical Planning and Inference, 89, 119-133.

Padgett, W. J.,Spurrier, J. D., 1990. Shewhart-type charts for percentiles of strength distributions. Journal of Quality Technology, 22, 283-288.

Seppala, T.,Moskowitz, H.,Plante, R.,Tang, J., 1995. Statistical process control via the subgroup bootstrap. Journal of Quality Technology, 27, 139-153.

Soliman, A. A., 2002. Reliability estimation in a generalized life-model with application to the Burr-XII. IEEE Transactions on Reliability, 51, 337-344.

Tadikamalla, P.R., 1980. A look at the Burr and related distributions. International Sta-

tistical Review, 48, 337-344.
Wang, B., 2008. Statistical Inference for the Burr Type XII Distribution. ACTA Mathematica Scientia, 28, 1103-1108.
Wang, F.K.,Keats, J.B., 1996. Maximum likelihood estimation of the Burr XII parameters with censored and uncensored data. Microelectronics and Reliability, 36, 359-362.
Wingo, D.R., 1983. Maximum likelihood methods for fitting the Burr type XII distribution to life test data. Biometrical Journal, 25, 77-84.
Wingo, D.R., 1993. Maximum likelihood methods for fitting the Burr type XII distribution to multiply (progressively) censored life test data. Metrika, 40, 203-210.
Zimmer, W.J.,Keats, J.B.,Wang, F.K., 1998. The Burr XII distribution in reliability analysis. Journal of Quality Technology, 30, 386-394.
Young, G. A., 1994. Bootstrap: More than a stab in the dark. Statistical Science, 9, 382-415.