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Distribution theory
Research Paper

The Gamma Modified Weibull Distribution

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Abstract

The Weibull distribution has been highlighted in recent decades for its wide use in important applied areas, Murthy (2004). The modified Weibull was proposed as a more flexible alternative for modeling data, Lai et al. (2003). Zografos and Balakrishnan (2009) pionnered a new class of distributions called the *gamma-G* with the advantage of having only one parameter to transform an arbitrary distribution. This simple fact allows us to explore a large number of skewed and non-skewed behaviors. In this paper, we present the main properties of the gamma modified Weibull distribution. We provide the moments, quantile function and other important measures. In addition, an application to a real data set demonstrates the usefulness of the new model.

Keywords: Gamma-G family · Gamma Weibull modified distribution · Hazard rate function · Maximum likelihood · Modified Weibull distribution · Weibull distribution

Mathematics Subject Classification: Primary 62E99

1. Introduction

The Weibull distribution is a highly known distribution due to its utility in modeling lifetime data where the hazard rate function (hrf) is monotone, Murthy et al. (2004). Moreover, it has the exponential and Rayleigh distributions as special models. Then, it may be adequate for fitting several types of data. When failure rates are modelled with the property of been monotone, the Weibull distribution is a good tool due to the fact that it may adapt negative and positive skewed density shapes. However, it can not be adopted to fit data sets with non-monotone hrf. This is a delicate issue because it is known that many data sets in survival analysis present bathtub and unimodal curves.

In 1980, Hjorth proposed a distribution with the special feature of being able to adapt failure rates with bathtub and unimodal shape curves, Hjorth (1980). This distribution had not the expected impact due to its complicated form. In recent years new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub hrf, Xie and Lie (1996). Among these, the exponentiated Weibull (EW) Mudholkar et al. (1995), additive Weibull Xie and Lie (1996) , extended Weibull, Xie et al. (2002), modified Weibull (MW) Lai et al. (2003), exponenciated Weibull, Mudholkar

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and Hudson (1996) and extended flexible Weibull Bebbington et al. (2007), among others. A good review of these models is presented in Pham and Lai (2007).

The cumulative distribution function (cdf) of the Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$ (for $x > 0$) is given by $H(x;\alpha,\beta) = 1 - \exp(-\alpha x^{\beta})$. The corresponding probability density function (pdf) is given by $h(x;\alpha,\beta) = \beta\alpha x^{\beta-1}\exp(-\alpha x^{\beta})$.

The Weibull distribution has been widely studied and used in several areas such as survival analysis (Kollia et al., 1996), reliability engineering (Kapur and Lamberson, 1977), weather forecasting (Stevens and Smulders, 1979), among others, since it is very flexible and its pdf curve changes drastically with the values of β.

The MW distribution (with parameters $\alpha > 0$, $\lambda > 0$ and $\beta > 0$), was pioneered by Sarhan and Zaindin (2009) as a generalization of the Weibull distribution with cdf given by Cordeiro et al. (2010a)

$$G(x;\alpha,\lambda,\beta) = 1 - \exp\{-\alpha x^{\beta}\exp(\lambda x)\}, \qquad x > 0, \tag{1}$$

where α is a scale parameter and λ and β are shape parameters. The associated pdf is given by

$$g(x,\alpha,\lambda,\beta) = \alpha x^{\beta-1}(\beta+\lambda x)\exp\{\lambda x - \alpha x^{\beta}\exp(\lambda x)\}, \qquad x > 0. \tag{2}$$

We shall write $Z \sim MW(\alpha,\lambda,\beta)$ to denote that the random variable Z has pdf (2).

Zografos and Balakrishnan proposed, in Zografos and Balakrishnan (2009), a family of univariate distributions generated by gamma random variables. For any baseline cdf $G(x)$, they defined the *gamma-G* distribution with pdf $f(x)$ and cdf $F(x)$ by

$$f(x) = \frac{1}{\Gamma(a)}[-\log\{1-G(x)\}]^{a-1}g(x), \tag{3}$$

and

$$F(x) = \frac{\gamma(a,-\log\{1-G(x)\})}{\Gamma(a)} = \frac{1}{\Gamma(a)}\int_0^{-\log\{1-G(x)\}} t^{a-1}\mathrm{e}^{-t}\,dt, \tag{4}$$

respectively, for $a > 0$, where $g(x) = dG(x)/dx$, $\Gamma(a) = \int_0^{\infty} t^{a-1}\mathrm{e}^{-t}\,dt$ and $\gamma(a,z) = \int_0^{z} t^{a-1}\mathrm{e}^{-t}\,dt$. The gamma-G distribution has the same parameters of the G distribution plus an additional shape parameter $a > 0$. Each new gamma-G distribution can be obtained from a specified G distribution. For $a = 1$, the G distribution is a basic exemplar of the gamma-G distribution with a continuous crossover towards cases with different shapes (for example, a particular combination of skewness and kurtosis). For example, Cordeiro et al. (2013) explores the properties of the gamma lomax distribution.

We define the *gamma modified Weibull* (GMW) density function by insterting (1) and (2) in equation (3). Then, we obtain a new four-parameter pdf given by

$$f(x) = \frac{1}{\Gamma(a)}\alpha x^{\beta-1}(\beta+\lambda x)\{\alpha x^{\beta}\exp(\lambda x)\}^{a-1}\exp\{\lambda x - \alpha x^{\beta}\exp(\lambda x)\}, \tag{5}$$

The corresponding cdf follows from equations (1) and (4) as

$$F(x) = \frac{\gamma(a,\alpha x^{\beta}\exp(\lambda x))}{\Gamma(a)} = \frac{1}{\Gamma(a)}\int_0^{\alpha x^{\beta}\exp(\lambda x)} t^{a-1}\mathrm{e}^{-t}\,dt. \tag{6}$$

Henceforth, if X is a random variable with pdf (5), we write $X \sim GMW(a, \alpha, \lambda, \beta)$. The hrf of X reduces to

$$\tau(x) = \frac{\alpha x^{\beta-1}(\beta+\lambda x)\{\alpha x^{\beta}\exp(\lambda x)\}^{a-1}\exp\{\lambda x - \alpha x^{\beta}\exp(\lambda x)\}}{\Gamma(a) - \gamma(a, \alpha x^{\beta}\exp(\lambda x))}. \tag{7}$$

The GMW distribution can also be applied in several areas such as the Weibull and MW distributions. Plots of the GMW pdf for selected parameter values are displayed in Figure 2. Note that skewed distributions and left and right heavy tail distributions have been obtained. Plots of the hrf are also displayed in Figure 3. We can note that, in general, the hrf plot is a non-decreasing function.

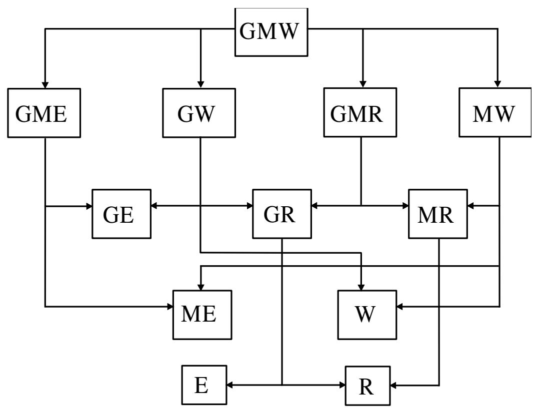

Figure 1. Relations between the GMW and other special models

We can obtain as special cases of the GMW distribution: the gamma Weibull (GW), MW, gamma exponential (GE), gamma modified exponential (GME), gamma modified Rayleigh (GMR), gamma exponential (GE), gamma Rayleigh (GR), modified Rayleigh (MR), modified exponential (ME), exponential (E), Rayleigh (R) and Weibull distributions. Figure 1 presents the relation between these distributions and the GMW model. We aim to prove that the GMW distribution may be used as a more general tool for modeling data that have some of the Weibull characteristic shapes.

We are interested on the mathematical properties of the proposed distribution. In sections 2, we derive a power series expression for the pdf. Asymptotic behaviours, quantile function (qf) and moments are presented in sections 3, 4 and 5, respectively. In section 6, we present the maximum likelihood estimation for the new model. Finally, we provide an application of the GMW model to a real data set in section 7.

2. Density expansion

Useful expanssions for the density and distribution functions can be derived using the concept of exponentiated distribution in Zografos and Balakrishnan (2009), whose properties have been widely studied in recent years. The generalized binomial coefficient with real arguments is defined by

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}.$$

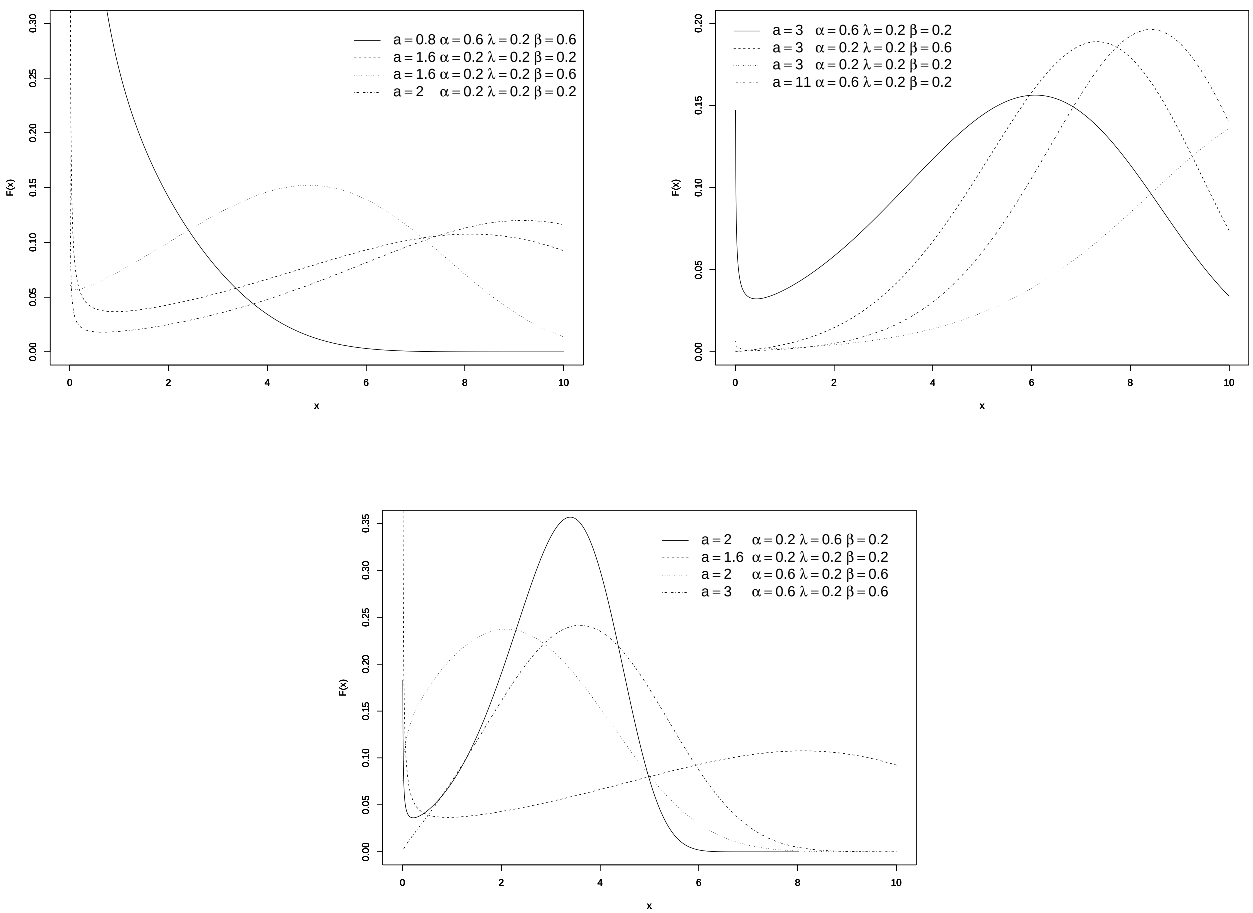

Figure 2. Plots of the GMW pdf for some parameter values

The quantity $[-\log\{1-G(x)\}]^{a-1}$ can be expanded as (For details here):

$$[-\log\{1-G(x)\}]^{a-1} = (a-1)\sum_{k=0}^{\infty}\binom{k+a-1}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}p_{j,k}}{a-1-j}\binom{k}{j}G(x)^{a+k-1}, \quad (8)$$

where the constants $p_{j,k}$ can be determined using the recursive formula:

$$p_{j,k} = \frac{1}{k}\sum_{m=1}^{k}\{k-m(j+1)\}\,c_m\,p_{j,k-m}, \quad (9)$$

for $j \geq 0$, $k = 1, 2, ...$, where $p_{j,0} = 1$ and $c_k = (-1)^{k+1}(k+1)^{-1}$ for $k \geq 1$.

Next, we can write $G(x)^{a+k-1}$ as

$$\begin{aligned} G(x)^{a+k-1} &= [1-\exp\{-\alpha x^{\beta}\exp(\beta x)\}]^{a+k-1} \\ &= \sum_{m=0}^{\infty}(-1)^m\binom{a+k-1}{m}\exp\{-\alpha m x^{\beta}\,\exp(\lambda x)\}. \quad (10) \end{aligned}$$

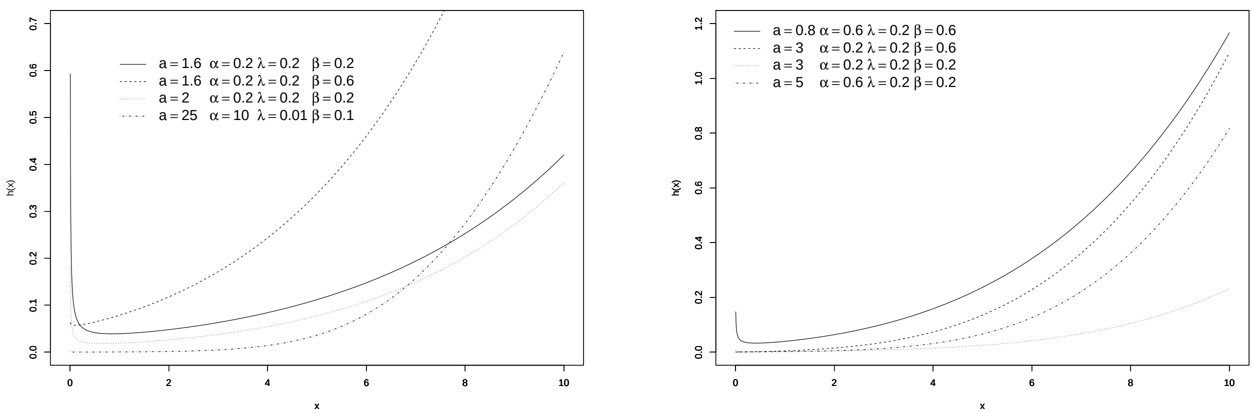

Figure 3. Plots of the hrf of X for some parameter values

Then, multiplying the last equation by the pdf of X, we have

$$\begin{aligned}
G(x)^{a+k-1}g(x) &= \sum_{m=0}^{\infty}(-1)^m \binom{a+k-1}{m} (\beta+\lambda x)\,\alpha x^{\beta-1}\exp\{-\alpha m x^{\beta}\exp(\lambda x)+\lambda x-\alpha x^{\beta}\exp(\lambda x)\} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\binom{a+k-1}{m} g(x;(m+1)\alpha,\lambda,\beta).
\end{aligned}$$

Further,

$$\begin{aligned}
f(x) &= \frac{1}{\Gamma(a-1)}\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{jk}}{a-1-j} \times \sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\binom{a+k-1}{m} g(x;(m+1)\alpha,\lambda,\beta) \\
&= \frac{1}{\Gamma(a-1)}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\binom{k+1-a}{k}\binom{a+k-1}{m}\frac{(-1)^{j+k+m}p_{jk}}{(a-1-j)(m+1)}\, g(x;(m+1)\alpha,\lambda,\beta).
\end{aligned}$$

Finally, we can write

$$f(x) = \sum_{m=0}^{\infty} w_m\, g(x;(m+1)\alpha,\lambda,\beta), \tag{11}$$

where

$$w_m = \frac{1}{\Gamma(a-1)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+m}p_{jk}}{(a-1-j)(m+1)}\binom{k}{j}\binom{k+1-a}{k}\binom{a+k-1}{m}. \tag{12}$$

Equation (11) reveals that the pdf of X can be expressed as an infinite weighted linear combination of MW densities. So, several structural properties of the GMW distribution can follow from (11) and those MW properties.

3. Asymptotes and shapes

Here, we present some properties of the GMW distribution such as the asymptotic behavior of the pdf (5) and hrf (7). We have the following results. For $x \to 0$,

- The pdf becomes

$$f(x) \sim \frac{(\beta+\lambda x)\alpha x^{\beta-1}}{\Gamma(a)}[1-\exp\{-\alpha x^{\beta}\exp(\lambda x)\}]^{a-1}\exp\{\lambda x-\alpha x^{\beta}\exp(\lambda x)\};$$

- The cdf becomes

$$F(x) \sim \frac{1}{\Gamma(a+1)}\{-\alpha x^{\beta}\exp(\lambda x)\}^{a};$$

- The hrf becomes

$$\tau(x) \sim \frac{1}{\Gamma(a)}[1-\exp\{-\alpha x^{\beta}\exp(\lambda x)\}]^{a-1};$$

For $x \to \infty$,

- The pdf becomes

$$f(x) \sim \frac{1}{\Gamma(a)}\alpha x^{\beta-1}(\beta+\lambda x)\{-\alpha x^{\beta}\exp(\lambda x)\}^{a-1}\exp\{\lambda x-\alpha x^{\beta}\exp(x\lambda)\};$$

- The cdf becomes

$$F(x) \sim \alpha x^{\beta-1}(\beta+\lambda x)\exp(\lambda x);$$

- The hrf becomes

$$\tau(x) \sim \frac{1}{\Gamma(a)}\{-\alpha x^{\beta}\exp(\lambda x)\}^{a-1}[-\exp\{-\alpha x^{\beta}\exp(\lambda x)\}].$$

For large values of x, the hrf and pdf have the same behaviour. For small values of x, the pdf behaves as its exponentiated version

4. Quantile function

Here, we derive the qf of the GMW distribution. We use the following result for $n \geq 1$

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i,$$

where

$$c_{n,i} = (i a_0)^{-1}\sum_{m=1}^{i}\{m(n+1)-i\}\, a_m\, c_{n,i-m},$$

and $c_{n,0} = a_0^n$. The coefficient $c_{n,i}$ can be obtained in terms of $c_{n,0}, \cdots, c_{n,i-1}$ and, therefore, from the quantities $a_0, a_1, \cdots, a_i$. Simulating the values of X follow by considering V as a gamma random variable with shape parameter a and scale parameter one. We have

$$X = G^{-1}\{1 - \exp(V)\},$$

where $G(x)$ denotes the MW cdf. Then, X follows the GMW distribution.

Further, the inverse function of $F(x)$, can be expressed as

$$F^{-1}(u) = G^{-1}[1 - \exp\{-Q^{-1}(a, 1-u)\}],$$

for $0 < u < 1$, where $Q^{-1}(a, u)$ is the inverse function of $Q(a, x) = 1 - \gamma(a, x)/\Gamma(a)$ (see, for more details) and $G^{-1}(u)$ is given by

$$G^{-1}(u) = \frac{\beta}{\lambda} W\left(\frac{\lambda\left\{-\alpha^{-1}\log(1-u)\right\}^{\frac{1}{\beta}}}{\beta}\right),$$

where the function $W(x)$ is the log product function (see, for further details). The qf of X reduces to

$$Q(u) = \frac{\beta}{\lambda} W\left(\frac{\lambda\left\{\alpha^{-1}Q^{-1}(a, 1-u)\right\}^{\frac{1}{\beta}}}{\beta}\right), \qquad 0 < u < 1, \tag{13}$$

Equation (13) allows us to obtain important quantiles by means of appropriate choices of u. For example, the median of X is given by

$$Md(X) = \frac{\beta}{\lambda} W\left(\frac{\lambda\left\{\alpha^{-1}Q^{-1}(a, 1/2)\right\}^{\frac{1}{\beta}}}{\beta}\right)$$

Expressions for the skweness and kurtosis may be obtained from (13). The Bowley's skewness is based on quartiles, Kenney and Keeping (1962) proposed the quantity

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)},$$

whereas the Moors' kurtosis Moors (1998) is based on octiles given by

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}.$$

5. Moments

The kth moment of X can be expressed as

$$\mathrm{E}(X^k) = \sum_{m=0}^{\infty} w_m \int_0^{\infty} x^k\, g(x; (m+1)\alpha, \lambda, \beta)\, \mathrm{d}x = \sum_{m=0}^{\infty} w_m\, \mathrm{E}(Y_{(m)}^k), \tag{14}$$

where $Y_{(m)}$ is a MW random variable with pdf $g(x; (m+1)\alpha, \lambda, \beta)$. The importance of this result is that the moments of the GMW distribution may be obtained as a linear combination of MW random variables moments. An infinite representation for the kth moment of the MW distribution is obtained in Cordeiro et al. (2008) as follows

$$\mathrm{E}(Y_{(m)}^k) \;=\; \sum_{i_1,\ldots,i_k=1}^{\infty} \frac{B_{i_1,\ldots,i_k}\,\Gamma(s_k/\beta\,+\,1)}{\{(\,m+1)\,\alpha\,\}^{s_k/\beta+1}}, \tag{15}$$

where

$$B_{i_1,\cdots,i_r} \;=\; b_{i_1}\ldots b_{i_k} \text{ and } s_k \;=\; i_1+\cdots+i_k,$$

and

$$b_i \;=\; \frac{(-1)^{i+1} i^{\,i-2}}{(i-1)!}\left(\frac{\lambda}{\beta}\right),$$

Equations (14) and (15) can be used to determinate the moments of the GMW distribution without any restrictions on its four parameters; for instance, the first moment becomes

$$\mathrm{E}(X) \;=\; \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} w_m\,\frac{b_i\,\Gamma(i/\beta\,+\,1)}{\{\,(\,m+1)\,\alpha\,\}^{i/\beta+1}}$$

6. Maximum likelihood estimation

Here, we examine the estimation of the parameters by maximum likelihood and perform inference for the GMW distribution. Let $X_1,\ldots,X_n$ be a random sample from $X \sim \mathrm{GMW}(a,\alpha,\lambda,\beta)$ with observed values $x_1,\ldots,x_n$ and let $\boldsymbol{\theta} = (a,\alpha,\lambda,\beta)^T$ be the vector of the parameters. The log-likelihood function for $\boldsymbol{\theta}$ reduces to

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n}\{\alpha x_i^{\beta-1}(\beta+\lambda x)\}+\{\lambda x-\alpha x_i^{\beta}\exp(\lambda x_i)\}+(a-1)\sum_{r=1}^{n}\log\left\{\alpha x_i^{\beta}\exp(\lambda x)\right\}-n\log\{\Gamma(a)\}. \tag{16}$$

The score vector is $U(\boldsymbol{\theta}) = (\partial\ell/\partial a, \partial\ell/\partial\alpha, \partial\ell/\partial k, \partial\ell/\partial b)^T$. The score components follow by differentiating (16) as

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial a} = \sum_{i=1}^{n}\log\{\alpha x_i^{\beta}\exp(\lambda x_i)\} - n\psi(a),$$

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial\alpha} = \sum_{i=1}^{n}\frac{a-\alpha\beta x_i\exp(\lambda x_i)}{\alpha},$$

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial\beta} = \sum_{i=1}^{n}\frac{\log(x_i)\,(\beta+\lambda x)\{a-\alpha x^{\beta}\exp(\lambda x)\}+1}{\beta+\lambda x},$$

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial\lambda} = \sum_{i=1}^{n} x_i\left[a+\alpha x_i^{\beta}\,\{-\exp(\lambda x_i)\}+\frac{1}{\beta+\lambda x_i}\right],$$

where $\psi(a)$ is the digama function. The maximum likelihood estimates (MLEs) of the parameters can be are obtained numerically by equating simultaneously these equations to zero. Then, we obtain the solutions by numerical methods due to the fact that these expressions do not have closed-form. For interval estimation and hypothesis tests on the model parameters, we require the observed information matrix $K(\theta) = \{-\kappa_{rs}\}$, where $\kappa_{rs} = \partial^2 l(\boldsymbol{\theta})/\partial\theta_r\,\partial\theta_s$ and $\theta_r, \theta_s \in \{a, \beta, \lambda, \alpha\}$, whose elements can be computed numerically.

7. Application

In this section, we present an application of the GMW to a real data set. We compare the results of fitting the GMW, the sub-models GW and MW, and the five-parameter beta modified Weibull (BMW) distribution proposed in Cordeiro et al. (2010b). We consider an uncensored data set corresponding to failure times for a particular windshield model including 88 observations that are classified as failed times of windshields. These data were previously study by Murthy et al. (2004). The data are:

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610
3.478	0.557	1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914
2.646	3.699	1.124	1.981	2.661	3.779	1.248	2.010	2.688	3.924	1.281
2.038	2.820	3.000	4.035	1.281	2.085	2.890	4.121	1.303	2.089	2.902
4.167	1.432	2.097	2.934	4.240	1.480	2.135	2.962	4.255	1.505	2.154
2.964	4.278	1.506	2.190	3.000	4.305	1.568	2.194	3.103	4.376	1.615
2.223	3.114	4.449	1.619	2.224	3.117	4.485	1.652	2.229	3.166	4.570
1.652	2.300	3.344	4.602	1.757	2.324	3.376	4.663			

Table 1 lists the MLEs for the model parameters of the fitted distributions. The model selection is carried out using the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics described in details by Chen and Balakirshnan (1995). In general, the smaller the values of the statistics W^* and A^*, the better the fitted model to the data.
All numerical evaluations such as the maximization of the likelihood function and the plots are carried out using the R software, see R Core Team (2012), particularly the calculation of the goodness of fit statistics were made using the R package detailed in Diniz and Barros (2013). Since the values of the W^* an A^* statistics are smaller for the GMW distribution compared with those values of the other models, we conclude that the new distribution is a very competitive model to the current data.

Further plots of the estimated pdfs of the GMW, GW and BMW models fitted to these data are displayed in Figure 4. They indicate that the GMW distribution is superior to the other distributions in terms of model fitting.

This indicates that the proposed GMW distribution produces better fit to the data than its sub-models and the BMW distribution.

Table 1. MLEs (standard erros in parentheses) and the statistics A^* and W^*

	Estimates					Statistic	
Distribution	α	β	λ	a	b	A^*	W^*
GMW	7.2255	0.0768	0.2079	13.5809	-	**0.4741**	**0.0626**
	(0.0452)	(0.0005)	(0.00008)	(0.1539)	-		
GW	0.0064	3.8247	-	0.4918	-	0.5659	0.0695
	(0.00001)	(0.0123)	-	(0.0002)	-		
MW	0.0637	1.1553	0.4932	-	-	0.7053	0.1067
	(0.00004)	(0.0064)	(0.0023)	-	-		
BMW	1.0940	0.1289	0.6004	4.9769	0.1823	0.4848	0.0683
	(0.0072)	(0.0032)	(0.0005)	(1.3616)	(0.0003)		

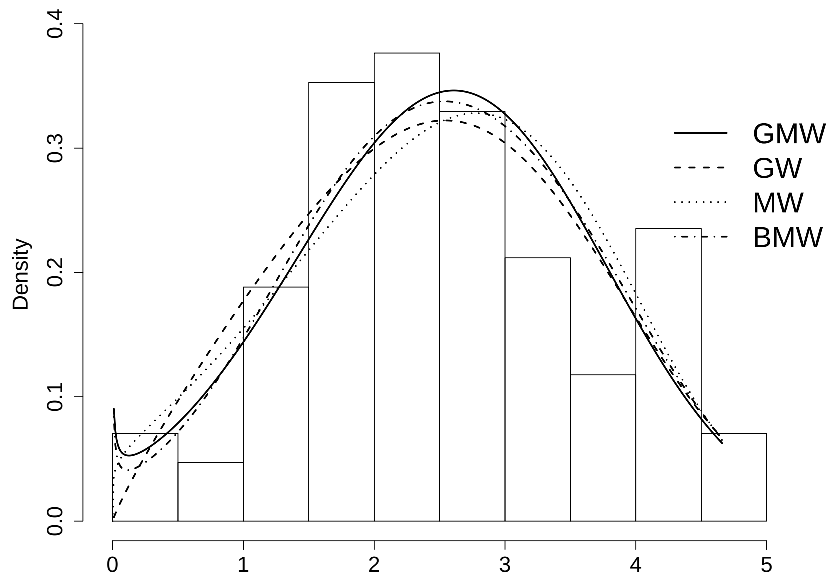

Figure 4. Fitted distributions

8. Concluding remarks

The modified Weibull (MW) distribution, a flexible extension of the Weibull distribution, is widely used to model lifetime data Sarhan and Zaindin (2009). We propose the gamma modified Weibull (GMW) distribution with the aim to extend the MW distribution introducing one extra parameter. This extension yields a broader class of hazard rate and density functions. We provide a mathematical treatment of the new distribution including expansions for the cumulative and density functions. We obtain the ordinary moments, quantile function, the asymptotes, shape and generating function. The estimation of the model parameters is approached by maximum likelihood. An application to a real data set indicates that the fit of the new model is superior to the fits of its main sub-models and the beta modified Weibull (BMW) model. We expect that the proposed model may be an interesting alternative model for a wider range of statistical research.

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References

Bebbington, M., Lai, C.D., Zitikis, R., 2007. A flexible weibull extension. Reliability Engineering & System Safety, 92(6), 719–726.

Chen, G., Balakirshnan, N., 1995. A general purpose approximate goodness-of-fit test. Journal of Quality Technology, 27(2), 154–161.

Cordeiro, G.M., Barreto-Souza, W., Santos, A.H.S., 2010a. The beta generalized exponential distribution. Journal of Statistical Computation and Simulation, 80(2), 159–172.

Cordeiro, G.M., Carrasco, J.M.F., Ortega, E.M.M., 2008. A generalized modified Weibull distribution for lifetime modeling. Computational Statistics and Data Analysis, 53, 450–462.

Cordeiro, G.M., Silva, G.O., Ortega, M.M.E., 2010b. The beta modified weibull distribution. Lifetime data analysis, 16(3), 409–430.

Cordeiro, G.M., Ortega, E.M.M, Popović, B.V., 2013. The gamma-lomax distribution. Journal of Statistical Computation and Simulation, 85(2), 305–319.

Diniz, P.R., Barros, C.R., 2013. Adequacy of probabilistic models and generation of pseudorandom numbers. R package.

Hjorth, U., 1980. A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates. Technometrics, 22(1), 99–107.

Kapur, K. C., Lamberson, L.R., 1977. Reliability in engineering design. New York, John Wiley and Sons, Inc. 605 p., 1.

Kenney, J.F., Keeping, E.S., 1962. Moving averages. Kenney JF. Mathematics of statics. Princeton NJ: Van Nostrand.

Kollia, G.D., Mudholkara, S., Srivastavab, D.K., 1996. A Generalization of the Weibull Distribution with Application to the Analysis of Survival Data. Journal of the American Statistical Association, 91(436), 1575–1583.

Lai, C.D., Mim, X., Murthy, D.N.P., 2003. A modified weibull distribution. Reliability, IEEE Transactions on, 52(1), 33–37.

Moors, J.J.A., 1998. A quantile alternative for kurtosis. Journal of the Royal Statistical Society, D 37, 25-32.

Mudholkar, G.S., Hutson, A.D., 1996. The exponentiated weibull family: Some properties and a flood data application. Communications in Statistics–Theory and Methods, 25(12), 3059–3083.

Mudholkar, G.S., Srivastava, D.K, Marshall, F., 1995. The exponentiated weibull family: A reanalysis of the bus-motor-failure data. Technometrics, 37(4), 436–445.

Murthy, D.N.P., Xie, M., Jiang, R., 2004. Weibull models, volume 505. John Wiley & Sons.

Sarhan, A.M., Zaindin, M., 2009. Modified weibull distribution. Applied Sciences, 11, 123–136

Stevens, M.J.M., Smulders, P.T., 1979. The estimation of the parameters of the weibull wind speed distribution for wind energy utilization purposes. Wind engineering, 3, 132–145.

Pham, H., Lai, C.D., 2007. On recent generalizations of the weibull distribution. IEEE Transactions on Reliability, 56(3), 454–458.

R Core Team, 2012. R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing. Vienna, Austria, 2012. Technical report, ISBN 3-900051-07-0.

Xie, M., Lai, C.D., 1996. Reliability analysis using an additive weibull model with bathtub-shaped failure rate function. Reliability Engineering & System Safety, 52(1), 87–93.

Xie, M., Tang, Y., Goh, T.N., 2002. A modified weibull extension with bathtub-shaped failure rate function. Reliability Engineering & System Safety, 76(3), 279–285.

Zografos, K., Balakrishnan, N., 2009. On families of beta-and generalized gamma-generated distributions and associated inference. Statistical Methodology, 6(4), 344–362.