

BAYESIAN INFERENCE
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Bayesian analysis for a class of beta mixed models

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Abstract

Generalized linear mixed models (GLMMs) encompass large class of statistical models, with a vast range of applications areas. GLMMs extend the linear mixed models allowing for different types of response variable. Three most common data types are continuous, counts and binary and standard distributions for these types of response variables are Gaussian, Poisson and binomial, respectively. Despite that flexibility, there are situations where the response variable is continuous, but bounded, such as rates, percentages, indexes and proportions. In such situations the usual GLMMs are not adequate because bounds are ignored and the beta distribution can be used. Likelihood and Bayesian inference for beta mixed models are not straightforward demanding a computational overhead. Recently, a new algorithm for Bayesian inference called INLA (Integrated Nested Laplace Approximation) was proposed. INLA allows computation of many Bayesian GLMMs in a reasonable amount time allowing extensive comparison among models. We explore Bayesian inference for beta mixed models by INLA. We discuss the choice of prior distributions, sensitivity analysis and model selection measures through a real data set. The results obtained from INLA are compared with those obtained by an MCMC algorithm and likelihood analysis. We analyze data from an study on a life quality index of industry workers collected according to a hierarchical sampling scheme. Results show that the INLA approach is suitable and faster to fit the proposed beta mixed models producing results similar to alternative algorithms and with easier handling of modeling alternatives. Sensitivity analysis, measures of goodness of fit and model choice are discussed.

Keywords: Hierarchical models · Bayesian inference · Beta law · Integrated Laplace approximation · Life quality.

Mathematics Subject Classification: Primary 62F15 · Secondary 65.

1. INTRODUCTION

There has been an increased interest in the class of Generalized Linear Mixed Models (GLMM). One possible reason for such popularity is that GLMM combine Generalized Linear Models (GLM) (Nelder and Wedderburn, 1972) with Gaussian random effects, adding flexibility to the models and accommodating complex data structures such as hierarchical, repeated measures, longitudinal, among others which typically induce extra

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variability and/or dependence.

GLMMs can also be viewed as natural extension of Mixed Linear Models (Pinheiro and Bates, 2000), allowing a wider class of probability distributions for response variables. Common choices are Gaussian for continuous data, Poisson and Negative Binomial for count data and Binomial for binary data. These three situations include the majority of applications within this class of models. Examples can be found in Breslow and Clayton (1993) and Molenberghs and Verbeke (2005).

Despite that flexibility, there are situations where the response variable is continuous and bounded above and below such as rates, percentages, indexes and proportions. In such situations the traditional GLMM based on the Gaussian distribution, is not adequate, since bounding is ignored. An approach that has been used to model this type of data is based on the beta distribution. The beta distribution is very flexible with density function that can display quite different shapes, including left or right skewness, symmetric, J-shape, and inverted J-shape (da Silva et al., 2011).

Regression models for independent and identically distributed beta variable were proposed by Paolino (2001), Kieschnick and McCullough (2003) and Ferrari and Cribari-Neto (2004). The basic assumption is that the response follows a beta law whose expected value is related to a linear predictor through a link function, similarly to GLM's. Cepeda (2001), Cepeda and Gamerman (2005) and Simas et al. (2010) extend the model regressing both, the mean and the dispersion parameters with covariates. Smithson and Verkuilen (2006) explores beta regression with an application to IQ data. Methods for likelihood based inference and model assessment are proposed by Espinheira et al. (2008b), Espinheira et al. (2008a) and Rocha and Simas (2011). Bias corrections for likelihood estimators are developed by Ospina et al. (2006), Ospina et al. (2011) and Simas et al. (2010). Branscum et al. (2007) adopts Bayesian inference to analyze virus genetic distances. Recently, Bonat et al. (2012) contrasted beta regression models with other approaches to model response variable on the unit interval, such that, Simplex, Kumaraswamy and Trans-Gaussian models. Results show that there is no overall prominent model.

The beta regression is implemented by `betareg` package (Cribari-Neto and Zeileis, 2010) for the R environment for statistical computing (R Development Core Team, 2012). Extended functionality is added for bias correction, recursive partitioning and latent finite mixture (Grün et al., 2012). Mixed and mixture models are further discussed by Verkuilen and Smithson (2012).

For non independent data, development have been proposed in times series analysis by McKenzie (1985), Grunwald et al. (1993) and Rocha and Cribari-Neto (2008). da Silva et al. (2011) use a Bayesian beta dynamic model for modeling and prediction of time series with an application to the Brazilian unemployment rates. Figueroa-Zúñiga et al. (2013) extend the beta model proposed by Ferrari and Cribari-Neto (2004) using a Bayesian approach. The authors considered two distributions for the random effects (Gaussian and t-Student) and several specifications for the prior distributions for parameters in the model.

Bonat et al. (2015) extend the beta model proposed by Ferrari and Cribari-Neto (2004) with the inclusion of Gaussian random effects, under a GLMM approach. Likelihood inference is based on two algorithms. The first uses the Laplace approximation to solve the integral in the likelihood function and the second uses an algorithm proposed by Lele et al. (2010) called data cloning. Authors analyzed two real data sets, with different structures for the random effects. Likelihood inference under GLMM is non-trivial because of presence random effects and several procedures have been proposed. Approximate likelihood methods are adopted by Breslow and Clayton (1993) and a Monte Carlo approach is adopted by Chen et al. (2002). Both come with a computational overhead. A popular approach is based upon a Bayesian framework using Markov Chain Monte Carlo (MCMC) algorithms with attempts to set non informative priors. Figueroa-Zúñiga et al. (2013) perform Bayesian

inference for beta mixed models using an MCMC algorithm. The Bayesian approach is attractive but requires specification of prior distributions, which is not straightforward, in particular for variance components.

Recently, Rue et al. (2009) introduced a novel numerical inference approach, the so-called Integrated Nested Laplace Approximation (INLA). INLA allows the computation of many Bayesian GLMMs in a reasonable amount of time, enabling for extensive comparisons of different models and prior distributions. Fong et al. (2010) used INLA for Bayesian analysis of several data sets and concluded that INLA is a very accurate algorithm and present results which help to guide the choice of prior distributions.

The main goal this paper is describe Bayesian inference for beta mixed models using INLA. We discuss the choice of prior distributions and measures of model comparisons. Results obtained from INLA are compared to those obtained using a Bayesian MCMC algorithm and a purely likelihood analysis. The modelling is illustrated through the analysis of a real dataset from a study on a life quality index of industry workers, with data collected according to a hierarchical sampling scheme. Additional care is given to choice of prior distributions for the precision parameter of the beta law.

The structure this paper is the follows. In Section 2, we define the Bayesian beta mixed model, Section 3 we describe the Integrated Nested Laplace Approximation (INLA). In Section 4 the model is introduced for the motivating example and the results of the analyses are presented. We close with concluding remarks in Section 5.

2. BAYESIAN BETA MIXED MODEL

Bayesian beta mixed regression extends the beta regression model, as proposed by Ferrari and Cribari-Neto (2004), by adding Gaussian distributed random effects to the linear predictor. Consider the response Y_{ij} from group $i = 1, \dots, N$ and replication $j = 1, \dots, n_i$. \mathbf{Y}_i is a $n_i - dimensional$ vector of measurements on the i^{th} group. Given a q -dimensional vector \mathbf{b}_i of random effects distributed as $N(\mathbf{0}, Q(\boldsymbol{\tau})^{-1})$, the responses Y_{ij} are conditionally independent with density function given by

$$\pi_i(y_{ij}|\mathbf{b}_i, \mu_{ij}, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu_{ij}\phi)\Gamma((1-\mu_{ij})\phi)} y_{ij}^{\mu_{ij}\phi-1} (1-y_{ij})^{(1-\mu_{ij})\phi-1}, \quad 0 < y < 1, \quad (1)$$

where $0 < \mu < 1$ is the mean of the response variable and $\phi > 0$ is a precision parameter. Let $g(\cdot)$ be a known link function with $g(\mu_{ij}) = \mathbf{x}_{ij}^T \boldsymbol{\beta} + \mathbf{z}_{ij}^T \mathbf{b}_i$, where \mathbf{x}_{ij} and \mathbf{z}_{ij} are vectors of covariates with dimensions p and q , respectively, and $\boldsymbol{\beta}$ is a p -dimensional vector of unknown regression parameters. Assume that $\mathbf{b}_i|Q(\boldsymbol{\tau}) \sim N(\mathbf{0}, Q(\boldsymbol{\tau})^{-1})$, where the precision matrix $Q(\boldsymbol{\tau})$ depends on parameters $\boldsymbol{\tau}$. The model specification is completed assuming prior distributions for all parameters in the model, say $\theta = (\boldsymbol{\beta}, \phi, \boldsymbol{\tau})$.

A flat improper prior is assumed for the intercept β_0 . All other components of $\boldsymbol{\beta}$ are assumed to be independent $N(0, \sigma^2)$ with fixed precision $\sigma^{-2} = 0.0001$. For the parameters in the precision matrix ($\boldsymbol{\tau}$) we follow an approach adopted by Fong et al. (2010) based on Wakefield (2009). The basic idea is to specify a range for the more interpretable marginal distribution of b_i and use this to derive the specification of the prior distributions. The approach is based on the result that if $b|\tau \sim N(0, \tau^{-1})$ and $\tau \sim Ga(a_1, a_2)$ then $b \sim t(0, a_2/a_1, 2a_1)$. To decide upon a prior, we define a range for a generic random effects b and specify the degrees of freedom, d , and then solve for a_1 and a_2 . The solution for a generic range, say $(-R, R)$, is $a_1 = d/2$ and $a_2 = R^2 d / (2(t_{1-(1-q)/2}^d))^2$. In linear mixed effects model, b is directly interpretable, while for beta models, it is more appropriate to think in terms of the marginal distribution of $exp(b)$. The prior distributions obtained this

way are flat. For more detailed description see Fong et al. (2010) section 4.2.

A flat $Ga(a_1 = 1, a_2 = 0.001)$ prior is chosen for ϕ as no result is known to aid its specification. The sensitivity to prior assumptions on the precision parameter of the beta law and on the precision parameters of the random effects is a potentially a delicate issue under beta mixed models. Figueroa-Zúñiga et al. (2013) considers several choices of prior distributions to ϕ but no sensitivity analysis is performed. The idea here is to specify this Gamma distribution as the default choice and then to assess the sensitivity.

3. INFERENCE, MODEL SELECTION AND SENSITIVITY

Bayesian inference on beta mixed models is not straightforward since the posterior distribution is not analytically available. Markov Chain Monte Carlo (MCMC) technique is the standard approach to fit such models (Figueroa-Zúñiga et al., 2013). In practice, this approach comes with a wide range of problems in terms of convergence and computational time. Moreover, the implementation itself can be problematic, especially for end users who might not be experts in programming. Software platforms for fitting generic random effects models via MCMC, include JAGS (Plummer, 2003), BayesX (Belitz et al., 2012) and WinBUGS (Lunn et al., 2000), among others.

Rue et al. (2013) is a newer tool for an end user based on the INLA (Integrated Nested Laplace Approximation) approach for Bayesian inference on latent Gaussian models with focus on the posterior marginal distributions (Rue et al., 2009). INLA replaces MCMC simulations by accurate deterministic approximations to posterior marginal distributions.

A computational implementation called `inla`, available at <http://www.r-inla.org>, allows the user to conveniently perform approximate Bayesian inference in latent Gaussian models. The `R` package `INLA` serves as an interface to `inla` routines and its usage is similar to the `glm` function in `R` (Roos and Held, 2011). Standard output provides marginal posterior densities for all parameters in the model and several measures of model goodness of fit.

The procedure of statistical analysis of a real data set, consists of specify the model, parameter estimation, comparisons among several models and evaluation results sensitivity, given the model specification. The second is tackled by INLA, and then the output includes several measures of model goodness of fit. The three more useful are, the Deviance Information Criterion (DIC), the log marginal likelihood (LML) and the conditional predictive ordinate (CPO), for more details, see Roos and Held (2011).

Roos and Held (2011) develop a general sensitivity measure based on the Hellinger distance to assess sensitivity of the posterior distributions with respect to changes on the prior distributions for precision parameters. Such methods is adopted here to assess the sensitivity to the choice of the prior distribution for ϕ and for the precision of the random effects. Following Roos and Held (2011), for a default θ_0 and a shifted θ prior value let

$$S(\theta_0, \theta) = \frac{H(\text{post}(\theta_0), \text{post}(\theta))}{H(\text{pri}(\theta_0), \text{pri}(\theta))} \quad (2)$$

denote the relative change on the posterior distribution with respect to changes on the prior distribution as measured by the Hellinger distance H , where $\text{pri}(\theta)$ is the prior distribution, $\text{post}(\theta)$ is the corresponding posterior distribution and

$$H(f, g) = \sqrt{1 - BC(f, g)}, \quad \text{where} \quad BC(f, g) = \int_{-\infty}^{\infty} \sqrt{f(u) \cdot g(u)} du.$$

The Hellinger distance is symmetric and measures the discrepancy between two densities

f and g . It takes a maximal value of 1 if BC is equal to 0 and is equal to 0 if and only if both densities are equal. The latter happens whenever the density f assigns probability 0 to every set to which the density g assigns a positive probability and vice versa. For more detailed description see Roos and Held (2011).

4. INCOME AND LIFE QUALITY OF BRAZILIAN INDUSTRY WORKERS

The Brazilian industry sector *worker's life quality index* (IQVT, acronym in Portuguese) is computed combining 25 indicators from eight thematic areas: housing, health, education, integral health and safety in the workplace, development of skills, value attributed to work, corporate social responsibility, stimulus to engagement and performance. The index is constructed following same premises as for the united nations human development index¹. The resulting values are in the unit interval and the closer to one the higher the worker's life quality in the industry.

A poll was conducted by Industry Social Service (*Serviço Social da Indústria - SESI*) in order to assess worker's life quality in the Brazilian industries. The survey included 365 companies on eight Brazilian federative units among the total of 26 states plus the Federal District. The data analysis considers two covariates related to the companies for which the impact on IQVT is of particular interest, namely, company average *income* and *size*. The first is given by the total of salaries divided by the number of workers expressing the capacity to fulfill individual basic needs such as food, health, housing and education. The second can be indirectly related to the capability of managing and providing quality of life.

The relevant question for the study and main goal here is to specify a suitable model to assess the influence of these two covariates on the IQVT. The federative unit where the company based is expected to influence the index considering varying local legislations, taxing and further economic and political conditions. This is accounted by including a random effect, regarding the eight states as a sample of the federative units.

Relations between the IQVT and the covariates income, size and with the states included in the survey are shown on Figure 1 which suggests all are potentially relevant. The income is expressed in logarithmic scale centered around the average.

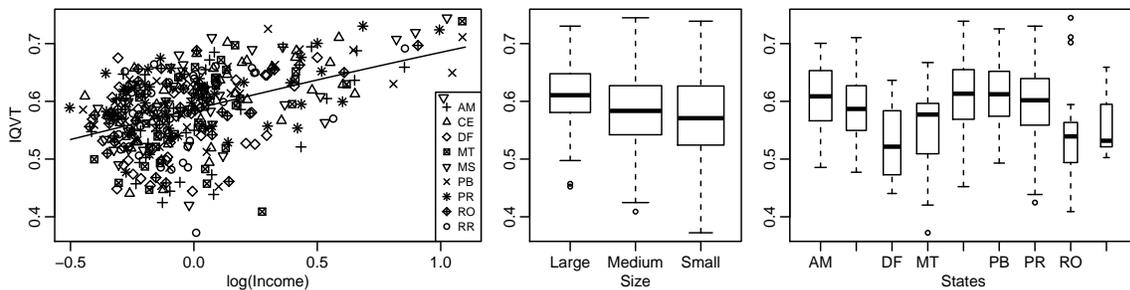


Figure 1. Scatterplot and boxplot relating IQVT with (centred) log income, company size and state.

¹<http://hdr.undp.org/en/humandev/>

Table 1. Posterior means, LML, DIC and CPO for the fitted models.

Parameter	Model 1	Model 2	Model 3	Model 4	Model 5
β_0	0.35	0.45	0.43	0.40	0.40
β_1		-0.11	-0.09	-0.07	-0.07
β_2		-0.16	-0.14	-0.13	-0.14
β_3			0.42	0.47	0.46
ϕ	53.92	56.44	72.16	93.37	93.28
τ_1^2				63.65	90.33
τ_2^2					532.73
τ_{12}					0.75
Goodness-of-fit					
LML	466.02	461.57	500.11	534.40	359.82
DIC	-941.11	-955.72	-1044.58	-1130.79	-1129.29
CPO	-1.29	-1.31	-1.43	-1.55	-1.55

The Bayesian beta random effects model for IQVT is given by

$$\begin{aligned}
 Y_{ij} | \mathbf{b}_i &\sim \text{Beta}(\mu_{ij}, \phi) \\
 g(\mu_{ij}) &= (\beta_0 + b_{i1}) + \beta_1 \text{Medium}_{ij} + \beta_2 \text{Small}_{ij} + (\beta_3 + b_{i2}) \text{Income}_{ij} \\
 \mathbf{b}_i &\sim N(\mathbf{0}, Q(\boldsymbol{\tau})^{-1}) \text{ with } Q(\boldsymbol{\tau}) = \begin{bmatrix} 1/\tau_1^2 & \tau_{12} \\ \tau_{12} & 1/\tau_2^2 \end{bmatrix}
 \end{aligned}$$

with $i = 1, \dots, 9$ indexing the federative units and $j = 1, \dots, n_i$ indexes the industry number within the i^{th} federative unit. The model is parametrized with β_0 associated with large size companies with differences β_1 and β_2 to the medium and small size, respectively. Random effects include an intercept b_{i1} and a slope b_{i2} associated with the covariate *income*. The vector of model parameters are the regression coefficients $(\beta_0, \beta_1, \beta_2, \beta_3)$, the random effects covariance parameters $(\tau_1^2, \tau_2^2, \tau_{12})$ and the precision parameter ϕ from beta law. The logit $g(\mu_{ij}) = \log\{\mu_{ij}/(1 - \mu_{ij})\}$ link function is used. The specification the Bayesian beta mixed model is completed by specifying the prior distributions for the model parameters. Following the remarks at Section 2 a flat improper prior is assumed for β_0 . All other components of β are assumed to be independent zero-mean $N(0, \sigma^2)$ with fixed precision $\sigma^{-2} = 0.001$. For the parameter ϕ we assumed a flat $Ga(a_1 = 1, a_2 = 0.0001)$ distribution. For the parameters indexing the random effects, we assumed that $Q \sim W_q(r, S)$, where $W_q(r, S)$ denote the Wishart distribution, r and S to be chosen as in the univariate case. Specifically, we assumed that $r = 5$ and a diagonal S with elements 0.001487 and 0.005, reducing to a $Ga(a_1 = 0.5, a_2 = 0.001487)$ when fitting the random intercept model.

A sequence of sub-models are defined in order to assess the effects of interest. Model 1 is a null model with just the intercept coefficient. Model 2 and 3 adds the covariates *size* and *income*, in this order. Model 4 and 5 adds random effects related to the *States* to the intercept and the *income* coefficient, respectively. The latter is the largest model considered here. A sequence of nested models are defined for comparison and detection of the relevant effects. Large size companies are considered as the baseline for the categorical covariate *size*. Table 1 shows the posterior means for the model parameters and model fitting measures given by the deviance information criterion (DIC), log marginal likelihood (LML) and conditional predictive ordinate (CPO), all obtained with INLA.

Results for models 1-3 confirm the relevance of the covariates. The increasing values for average posterior of ϕ , from 53.92 on model 1 to 72.16 on model 3, confirms further explanation of the data variability by the covariates. The random intercept clearly improves

Table 2. Comparison on the intervals obtained by different methods.

Parameter	MCMC.in.Profile	MCMC.in.INLA	Profile.in.INLA
β_0	0.9415	0.9435	0.9487
β_1	0.9457	0.9471	0.9490
β_2	0.9496	0.9509	0.9491
β_3	0.9465	0.9471	0.9491
ϕ	0.9491	0.9510	0.9480
τ_1^2	0.9446	0.9433	0.9511

the model fit, capturing the variability of the IQVT among the states. The addition of random slope did not prove relevant. All model fitting measures favors model 4 for which we report further analysis.

Figure 2 shows posterior distributions from INLA and a MCMC output from JAGS running three chains of 500,000 samples with a burn-in of 10,000 interactions and saving one of each 100 simulations. We also compared INLA results with likelihood point estimates and profile intervals. Figure 2 suggests that all approaches produced similar results. This is also assured by the results in Table 2 where the second and third columns provide the proportion of MCMC samples which falls into the profile likelihood interval and credibility intervals from INLA, respectively. The last column is the probability between the limits of the profile likelihood interval computed on the INLA marginal distribution. These results indicates the flat prior has little impact on the respective posterior distribution. The INLA and MCMC algorithms are similar in the inferential purposes but INLA is much faster and easier to use than MCMC.

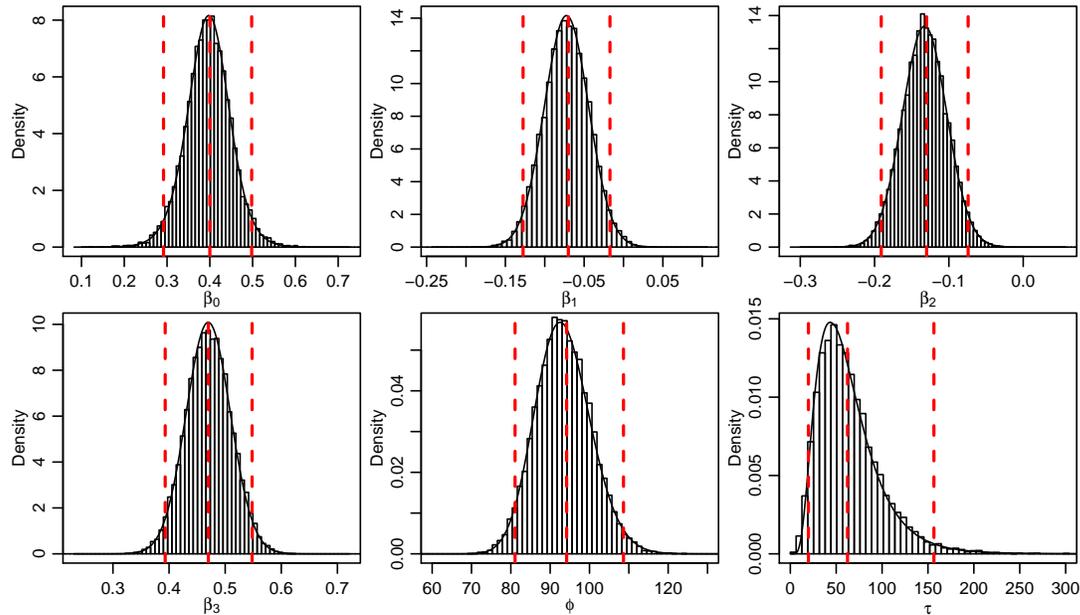


Figure 2. Marginal posterior densities from INLA and MCMC outputs. The vertical dashed lines correspond to point estimates and profile likelihood intervals.

We conclude the analysis assessing sensitivity to the choice the prior distributions. Following Roos and Held (2011), we investigate the sensitivity by measuring the Hellinger distance and focusing only on the parameters ϕ and τ since the choice of prior is standard for the regression coefficients β 's. To assess sensitivity we choose a set of prior distributions with determined Hellinger distance from the default prior, refit the model under those priors and compute the Hellinger distances between the corresponding posterior distributions. For example, by choosing $\phi \sim Ga(b1 = 1, b2 = 0.0135)$ the Hellinger distance from the

Table 3. Hellinger distances between the prior and posterior distributions from the ones obtained with the default prior.

Priori	HL Prior	HL Post	$S(post, pri)$
$\phi \sim Ga(b_1 = 1, b_2 = 0.0135)$	0.1058	0.0100	0.0945
$\phi \sim Ga(b_1 = 1, b_2 = 0.0178)$	0.2005	0.0200	0.0998
$\phi \sim Ga(b_1 = 1, b_2 = 0.0242)$	0.3005	0.0346	0.1153
$\phi \sim Ga(b_1 = 1, b_2 = 0.0338)$	0.4006	0.0583	0.1455
$\phi \sim Ga(b_1 = 1, b_2 = 0.050)$	0.5046	0.0975	0.1932
$\phi \sim Ga(b_1 = 1, b_2 = 0.0765)$	0.6004	0.1628	0.2711
$\tau \sim Ga(b_1 = 0.5, b_2 = 0.00225)$	0.1030	0.0100	0.0971
$\tau \sim Ga(b_1 = 1, b_2 = 0.0035)$	0.2086	0.0245	0.1174
$\tau \sim Ga(b_1 = 1, b_2 = 0.0055)$	0.3085	0.0458	0.1485
$\tau \sim Ga(b_1 = 1, b_2 = 0.0088)$	0.4017	0.0812	0.2022
$\tau \sim Ga(b_1 = 1, b_2 = 0.016)$	0.5031	0.1543	0.3066
$\tau \sim Ga(b_1 = 1, b_2 = 0.033)$	0.6022	0.2827	0.4694

Table 4. Summaries of the fitted models with Hellinger distances of 0.6 from the default prior for parameters ϕ and τ .

	Default	Std. Err	$H(\phi_D, \phi_{0.6}) = 0.6$	Std. Err	$H(\tau_D, \tau_{0.6}) = 0.6$	Std. Err
β_0	0.3965	0.0520	0.3966	0.0518	0.3950	0.0607
β_1	-0.0724	0.0282	-0.0725	0.0287	-0.0719	0.0282
β_2	-0.1327	0.0299	-0.1327	0.0304	-0.1320	0.0299
β_3	0.4700	0.0396	0.4695	0.0403	0.4718	0.0396
ϕ	93.3746	7.0019	90.1881	6.7830	93.5137	6.9848
τ_1^2	63.6521	34.2202	64.8726	34.3975	41.8625	20.8770

default $\phi_0 \sim Ga(a_1 = 1, a_2 = 0.001)$ prior is $H(Ga(a_1 = 1, a_2 = 0.001), Ga(b_1 = 1, b_2 = 0.0135)) = 0.1058$ whereas the Hellinger distance between the posteriors is 0.0100 and $S(\phi_0, \phi) = 0.0945$, i.e, the distance between the posterior distributions is only about one tenth of the distance between the prior distributions reflecting a major effect of the that and a little impact of choosing either prior. Table 3 shows the hyperparameters obtained for priors with Hellinger distances from about 0.1 to 0.6 and the corresponding Hellinger distances between priors, posteriors and $S(\cdot, \cdot)$. The distributions are plotted in Figure 3.

The results show that the models are more sensitive to the choice of prior for the parameter τ . For the parameter ϕ even with the rather large distance of 0.6 between prior distributions the corresponding distance between the posterior distributions is substantially reduced to 0.1628. The same distance between priors for the parameter τ still reduces to 0.2827. The posterior distributions in Figure 3 are similar for all prior distributions. Comparatively, the parameter τ is more sensitive to the choice of prior distribution, however still with similar posterior distributions even with large difference between priors.

Table 4 compare summary results of models with default prior and with the largest Hellinger distance from the default prior. For ϕ the posterior mean changed from 93.37 to 90.18, a difference is only 3.53% whereas for τ they change from 63.65 to 41.86 with a difference of 52.05%. Despite such differences, the practical conclusions on effects of relevance are unchanged since the changes are very small for the regression parameters. The relevance of the random effects in the model remains important.

5. CONCLUSION

This paper reports results of a Bayesian analysis of beta mixed models comparing results obtained with the INLA method with the ones obtained with an MCMC algorithm and

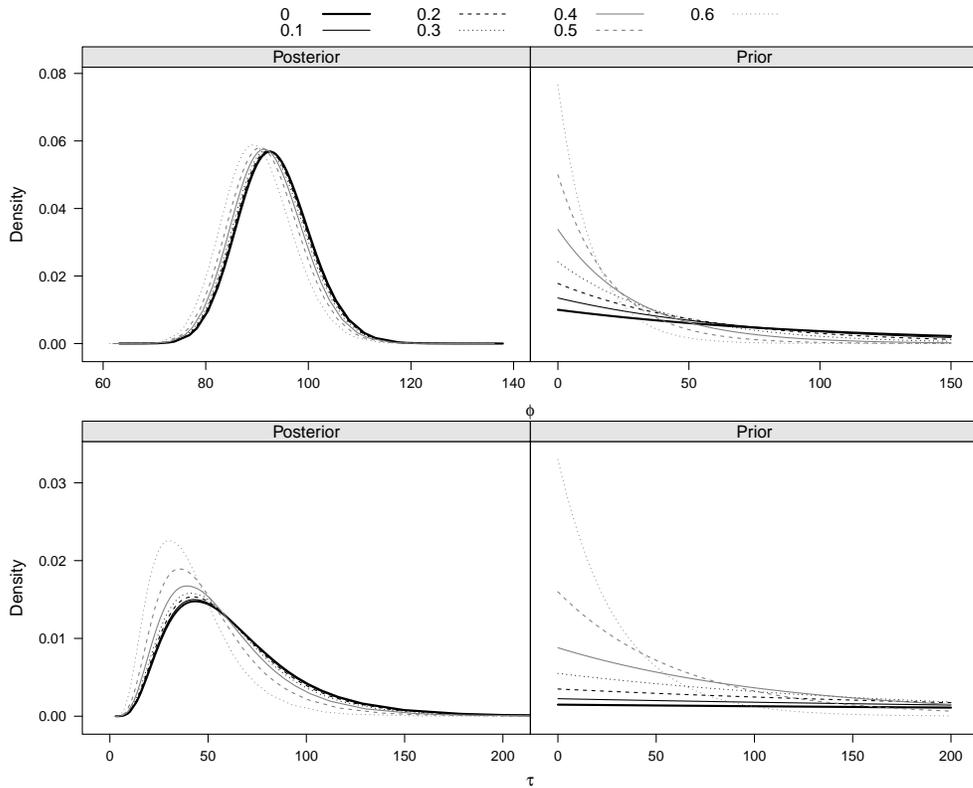


Figure 3. Posterior (left) and prior (right) distributions with Hellinger distances to the default prior ranging from 0.1 to 0.6 for the parameters ϕ (top) and τ (bottom).

purely likelihood analysis. Emphasis is placed on the specification and sensitivity of priors for the beta precision parameter and the precision of the random effects.

Results of the analysis of the index of life quality for the worker’s on the Brazilian industrial sector indicates company size and average income are both relevant for the quality of life, as well as the effect of the states captured by adding a random intercept to the regression model. The analysis consisted of fitting several models with one final model chosen according to three criteria of model comparisons – LML, DIC and CPO. All criteria points to the same model choice. Summary results obtained with INLA are similar with the ones obtained with MCMC and likelihood analysis showing the substantial gain in the computational burden makes INLA an attractive choice for inference which allowing for several modeling alternatives to be investigated.

The sensitivity analysis was conducted for the precision parameters in the Bayesian beta mixed model using the Hellinger divergence as a measure of the distance between prior and posterior distributions. Our results show that the beta precision parameter ϕ is insensitive to the choice of prior. Slightly more sensitive is the parameter τ related to the random effects, but the overall results and conclusions remains unchanged for the alternative priors.

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