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Distribution Theory
Research Paper

A new Bivariate Distribution with Log-Exponentiated Kumaraswamy Marginals

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Abstract

In this paper we introduce a new bivariate log-exponentiated Kumaraswamy (Blog-EK) distribution. This new singular distribution has univariate log-EK distributions marginals. Some properties are studied; parameter estimation using maximum likelihood estimators (MLEs) of the unknown parameters cannot be obtained in explicit forms. A real data set and simulation study are used to obtain the parameters estimation, and confidence intervals to study the behavior of the parameters.

Keywords: Generalized bivariate model · Joint probability density function · Conditional probability · Maximum likelihood estimators · Moment estimators · Fisher information matrix.

Mathematics Subject Classification: Primary 62H12 · Secondary 62E20.

1. Introduction

Recently Lemonte et al. (2013) introduced a three-parameter log-EK distribution by exponentiating the Kumaraswamy distribution, and it can be used to analyze several life time data. The log-EK distribution which extends the generalized exponential Gupta and Kundu (1999) and double generalized exponential Barreto–Souza et al. (2010) distributions.

The main aim of this paper is to introduce a new bivariate log-EK distribution, whose marginals are log-EK distributions. This new five-parameter Blog-EK distribution is obtained using a method similar to that used to obtain the Marshall-Olkin bivariate exponential model, Marshall and Olkin (1967), Sarhan and Balakrishnan (2007) and bivariate generalized exponential model of Kundu and Gupta (2009) The proposed Blog-EK distribution is constructed from three independent log-EK distributions using a maximization process. it is not easy to compute the maximum likelihood estimators of the four unknown parameters. Computation of the parameters involves solving a one dimensional optimization problem, when a one parameter is known. The generation of random samples from the Blog-EK is quite straight forward, which makes it very convenient to perform the

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simulation experiments.

Several properties of this new Blog-EK distribution are established. The joint probability density function (PDF) and the joint cumulative distribution function (CDF) are expressed in explicit forms. The marginals of the Blog-EK distribution are univariate log-EK distribution. the MLEs of the unknown parameters can be obtained by solving four nonlinear equations. Simulations are performed for illustrative purpose.

The rest of the paper is organized as follows. In section 2, we define the Blog-EK distribution and provide different properties of the proposed model. In section 3, we introduced the marginal and joint moment generating function. In section 4, we discuss the maximum likelihood estimation procedure of the unknown parameters. In section 5, we present the simulation and one data analysis results. Finally, a conclusion for the results given in section 6.

2. The New Bivariate Log-exponentiated Kumaraswamy Distribution

It is assumed that the univariate log-EK distribution with parameters $\alpha, \lambda, \gamma > 0$ has the PDF and CDF given by

$$f(x; \alpha, \lambda, \gamma) = \alpha\lambda\gamma e^{-x}(1 - e^{-x})^{\gamma-1}(1 - (1 - e^{-x})^{\gamma})^{\lambda-1}(1 - (1 - (1 - e^{-x})^{\gamma})^{\lambda})^{\alpha-1} \quad (1)$$

The corresponding CDF is

$$F(x; \alpha, \lambda, \gamma) = (1 - (1 - (1 - e^{-x})^{\gamma})^{\lambda})^{\alpha} \quad (2)$$

2.1 The Joint Cumulative Distribution Function

Suppose $U_1 \sim$ log-EK$(\alpha_1, \lambda, \gamma)$, $U_2 \sim$ log-EK$(\alpha_2, \lambda, \gamma)$ and $U_3 \sim$ log-EK$(\alpha_3, \lambda, \gamma)$ and they are independently distributed. Define $X_1 = max(U_1, U_3)$, and $X_2 = max(U_2, U_3)$.Then, the bivariate vector (X_1, X_2) has Blog-EK distribution with parameters $\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma$, and will be denoted by Blog-EK$(\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma)$.

We now study the joint distribution of the random variables X_1 and X_2. The following lemma gives the joint CDF of the Blog-EK$(\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma)$.

Lemma 2.1 The joint CDF of X_1 and X_2 is

$$\begin{aligned} F_{\text{Blog-EK}}(x_1, x_2) &= (1 - (1 - (1 - e^{-x_1})^{\gamma})^{\lambda})^{\alpha_1}(1 - (1 - (1 - e^{-x_2})^{\gamma})^{\lambda})^{\alpha_2} \\ &\times (1 - (1 - (1 - e^{-z})^{\gamma})^{\lambda})^{\alpha_3} \end{aligned} \quad (3)$$

where $z = min(x_1, x_2)$

Proof. Since

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$$

we have

$$\begin{aligned} F(x_1, x_2) &= P(max(U_1, U_3) \leq x_1, (max(U_2, U_3) \leq x_2) \\ &= P(U_1 \leq x_1, U_3 \leq x_1, U_2 \leq x_2, U_3 \leq x_2) \\ &= P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq min(x_1, x_2)) \end{aligned}$$

As $U_i (i = 1, 2, 3)$ are mutually independent, we readily obtain

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) &= P(U_1 \leq x_1)P(U_2 \leq x_2)P(U_3 \leq min(x_1, x_2)) \\ &= F_{\text{log-EK}}(x_1; \alpha_1, \lambda, \gamma)F_{\text{log-EK}}(x_2; \alpha_2, \lambda, \gamma)F_{\text{log-EK}}(z; \alpha_3, \lambda, \gamma) \end{aligned} \tag{4}$$

Substituting from (2) into (4), we obtain (3), which completes the proof of the lemma.2.1.

Corollary 2.2 The joint CDF of the Blog-EK$(\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma)$ in (4) can also be written as

$$F_{X_1,X_2}(x_1, x_2) = \begin{cases} F_{\text{log-EK}}(x_1; (\alpha_1 + \alpha_3), \lambda, \gamma)F_{\text{log-EK}}(x_2; \alpha_2, \lambda, \gamma) & \text{if } x_1 < x_2 \\ F_{\text{log-EK}}(x_1; \alpha_1, \lambda, \gamma)F_{\text{log-EK}}(x_2; (\alpha_2 + \alpha_3), \lambda, \gamma) & \text{if } x_2 < x_1 \\ F_{\text{log-EK}}(x; (\alpha_1 + \alpha_2 + \alpha_3), \lambda, \gamma) & \text{if } x_1 = x_2 = x \end{cases}$$

2.2 The Joint Probability Density Function

The following theorem gives the joint PDF of the Blog-EK

Theorem 2.3 If the joint CDF of (X_1, X_2) is as in (3), the joint PDF of (X_1, X_2) is given by

$$f(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } x_1 < x_2 \\ f_2(x_1, x_2) & \text{if } x_2 < x_1 \\ f_3(x_1, x_2) & \text{if } x_1 = x_2 = x \end{cases} \tag{5}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= (\alpha_1 + \alpha_3)\lambda^2\gamma^2 e^{-x_1}(1 - e^{-x_1})^{\gamma-1}(1 - (1 - e^{-x_1})^{\gamma})^{\lambda-1} \\ &\times (1 - (1 - (1 - e^{-x_1})^{\gamma})^{\lambda})^{\alpha_1+\alpha_3-1}\alpha_2 e^{-x_2}(1 - e^{-x_2})^{\gamma-1} \\ &\times (1 - (1 - e^{-x_2})^{\gamma})^{\lambda-1}(1 - (1 - (1 - e^{-x_2})^{\gamma})^{\lambda})^{\alpha_2-1} \end{aligned} \tag{6}$$

$$\begin{aligned} f_2(x_1, x_2) &= \alpha_1\lambda^2\gamma^2 e^{-x_1}(1 - e^{-x_1})^{\gamma-1}(1 - (1 - e^{-x_1})^{\gamma})^{\lambda-1}(\alpha_2 + \alpha_3) \\ &\times (1 - (1 - (1 - e^{-x_1})^{\gamma})^{\lambda})^{\alpha_1-1}e^{-x_2}(1 - e^{-x_2})^{\gamma-1} \\ &\times (1 - (1 - e^{-x_2})^{\gamma})^{\lambda-1}(1 - (1 - (1 - e^{-x_2})^{\gamma})^{\lambda})^{\alpha_2+\alpha_3-1} \end{aligned} \tag{7}$$

$$\begin{aligned} f_3(x, x) &= \alpha_3\lambda\gamma e^{-x}(1 - e^{-x})^{\gamma-1}(1 - (1 - e^{-x})^{\gamma})^{\lambda-1} \\ &\times (1 - (1 - (1 - e^{-x})^{\gamma})^{\lambda})^{\alpha_1+\alpha_2+\alpha_3-1} \end{aligned} \tag{8}$$

PROOF. Let us first assume that $x_1 < x_2$. In this case, $F_{\text{Blog-EK}}(x_1, x_2)$ in (3) becomes

$$F_1(x_1, x_2) = (1-(1-(1-e^{-x_1})^{\gamma})^{\lambda})^{\alpha_1+\alpha_3}(1-(1-(1-e^{-x_2})^{\gamma})^{\lambda})^{\alpha_2}$$

Then, upon differentiation, we obtain the expression of $f_{\text{Blog-EK}}(x_1, x_2) = \frac{\partial^2 F_1(x_1,x_2)}{\partial x_1 \partial x_2}$ to be $f_1(x_1, x_2)$ given in (6). Similarly, we find the expression of $f_{\text{Blog-EK}}(x_1, x_2)$ to be $f_2(x_1, x_2)$ given in (7) when $x_2 < x_1$. But, $f_3(x, x)$ can not be derived in a similar way. For this reason, we use the identity

$$\int_0^{\infty}\int_0^{x_2} f_1(x_1, x_2)dx_1dx_2 + \int_0^{\infty}\int_0^{x_1} f_2(x_1, x_2)dx_2dx_1 + \int_0^{\infty} f_3(x, x)dx = 1$$

One can verify that

$$I_1 = \int_0^{\infty}\int_0^{x_2} f_1(x_1, x_2)dx_1dx_2 \quad \text{and} \quad I_2 = \int_0^{\infty}\int_0^{x_1} f_2(x_1, x_2)dx_2dx_1$$

$$\begin{aligned} I_1 = &\int_0^{\infty} \alpha_2\lambda\gamma e^{-x_2}(1-e^{-x_2})^{\gamma-1}(1-(1-e^{-x_2})^{\gamma})^{\lambda-1} \times (1-(1-(1-e^{-x_2})^{\gamma})^{\lambda})^{\alpha_2-1} \\ &\times \int_0^{x_2} (\alpha_1+\alpha_3)\lambda\gamma e^{-x_1}(1-e^{-x_1})^{\gamma-1}(1-(1-e^{-x_1})^{\gamma})^{\lambda-1} \\ &\times (1-(1-(1-e^{-x_1})^{\gamma})^{\lambda})^{\alpha_1+\alpha_3-1}dx_1dx_2 \end{aligned}$$

then

$$\begin{aligned} I_1 = &\int_0^{\infty} \alpha_2\lambda\gamma e^{-x_2}(1-e^{-x_2})^{\gamma-1}(1-(1-e^{-x_2})^{\gamma})^{\lambda-1} \\ &\times (1-(1-(1-e^{-x_2})^{\gamma})^{\lambda})^{(\alpha_1+\alpha_2+\alpha_3)-1}dx_2 \end{aligned} \tag{9}$$

similarly

$$\begin{aligned} I_2 = &\int_0^{\infty} \alpha_1\lambda\gamma e^{-x_1}(1-e^{-x_1})^{\gamma-1}(1-(1-e^{-x_1})^{\gamma})^{\lambda-1} \\ &\times (1-(1-(1-e^{-x_1})^{\gamma})^{\lambda})^{(\alpha_1+\alpha_2+\alpha_3)-1}dx_1 \end{aligned} \tag{10}$$

From (9) and (10), we then get

$$\begin{aligned} \int_0^{\infty} f_3(x, x)dx = &\int_0^{\infty} \alpha_3\lambda\gamma e^{-x}(1-e^{-x})^{\gamma-1}(1-(1-e^{-x})^{\gamma})^{\lambda-1} \\ &\times (1-(1-(1-e^{-x})^{\gamma})^{\lambda})^{(\alpha_1+\alpha_2+\alpha_3)-1}dx \end{aligned}$$

$$\begin{aligned} f_3(x, x)dx = &\alpha_3\lambda\gamma e^{-x}(1-e^{-x})^{\gamma-1}(1-(1-e^{-x})^{\gamma})^{\lambda-1} \\ &\times (1-(1-(1-e^{-x})^{\gamma})^{\lambda})^{(\alpha_1+\alpha_2+\alpha_3)-1} \end{aligned} \tag{11}$$

This completes the proof of the theorem.

Corollary 2.4 The joint probability density function of X_1 and X_2 as provided in Theorem 2.3, can also be expressed in the following form for $z = min(x_1, x_2)$, and for $f_1(.,.), f_2(.,.)$ same as defined in (6) and (7) respectively;

$$f(x_1, x_2) = \frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} f_a(x_1, x_2) + \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_s(x) \tag{12}$$

where

$$f_a(x_1, x_2) = \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + \alpha_2} \begin{cases} f_{\text{log-EK}}(x_1; \alpha_1 + \alpha_3, \lambda, \gamma) f_{\text{log-EK}}(x_2; \alpha_2, \lambda, \gamma) & \text{if } x_1 < x_2 \\ f_{\text{log-EK}}(x_1; \alpha_1, \lambda, \gamma) f_{\text{log-EK}}(x_2; \alpha_2 + \alpha_3, \lambda, \gamma) & \text{if } x_2 < x_1 \end{cases}$$

and

$$f_s(x) = f_{\text{log-EK}}(x; \alpha_1 + \alpha_2 + \alpha_3, \lambda, \gamma)$$

Clearly, here $f_a(x_1, x_2)$ and $f_s(x)$ are the absolute continuous part and singular part respectively.

2.3 Marginal and Conditional Probability Density Functions

In this section, we derive the marginal density functions of X_i and the conditional density functions of $X_i|X_j, (i \neq j = 1, 2)$. We also present the joint moment generating function of X_1 and X_2.

Theorem 2.5 The marginal pdf of $X_i (i = 1, 2)$ is given by

$$\begin{aligned} f_{X_i}(x_i) &= (\alpha_i + \alpha_3)\lambda\gamma e^{-x_i}(1 - e^{-x_i})^{\gamma-1}(1 - (1 - e^{-x_i})^{\gamma})^{\lambda-1} \\ &\times (1 - (1 - (1 - e^{-x_i})^{\gamma})^{\lambda})^{\alpha_i+\alpha_3-1} \end{aligned} \tag{13}$$

Proof. The marginal pdf of X_i can be derived from the marginal cumulative distribution function of X_i , say $F_{X_i}(x_i)$, as follows:

$$F_{X_i}(x_i) = P(X_i \leq x_i) = P(max(U_i, U_3) \leq x_i) = P(U_i \leq x_i, U_3 \leq x_i)$$

and since U_i is independent of U_3, we simply have

$$F_{X_i}(x_i) = (1 - (1 - (1 - e^{-x_i})^{\gamma})^{\lambda})^{\alpha_i+\alpha_3}$$

from which we readily derive the pdf of X_i, $f(x_i) = \frac{\partial F(x_i)}{\partial x_i}$, given in (13).

Corollary 2.6 Let $(X_1, X_2) \sim$ Blog-EK$(\alpha_1 + \alpha_2 + \alpha_3, \lambda, \gamma)$, then

(1) $X_1 \sim$ log-EK$(x_1; \alpha_1 + \alpha_3, \lambda, \gamma)$
(2) $X_2 \sim$ log-EK$(x_2; \alpha_2 + \alpha_3, \lambda, \gamma)$
(3) $X_3 = max(X_1, X_2) \sim$ log-EK$(x_3; \alpha_1 + \alpha_2 + \alpha_3, \lambda, \gamma)$

2.4 Conditional Probability Density Functions

Having obtained the marginal probability density functions of X_1 and X_2 we can now derive the conditional probability density functions as presented in the following theorem.

Theorem 2.7 The conditional pdf of X_i, given $X_j = x_j$, denoted by $f_{i|j}(x_i|x_j), (i \neq j = 1, 2)$, is given by

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & \text{if } x_i < x_j \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & \text{if } x_j < x_i \\ f^{(3)}_{X_i|X_j}(x_i|x_j) & \text{if } x_i = x_j = x \end{cases}$$

where

$$f^{(1)}_{X_i|X_j}(x_i|x_j) = \frac{(\alpha_1+\alpha_3)\alpha_2\lambda\gamma e^{-x_i}(1-e^{-x_i})^{\gamma-1}(1-(1-e^{-x_i})^{\gamma})^{\lambda-1}(1-(1-(1-e^{-x_i})^{\gamma})^{\lambda})^{\alpha_1+\alpha_3-1}}{(\alpha_2+\alpha_3)(1-(1-(1-e^{-x_j})^{\gamma})^{\lambda})^{\alpha_3}}$$

$$f^{(2)}_{X_i|X_j}(x_i|x_j) = \alpha_1\lambda\gamma e^{-x_i}(1-e^{-x_i})^{\gamma-1}(1-(1-e^{-x_i})^{\gamma})^{\lambda-1}(1-(1-(1-e^{-x_i})^{\gamma})^{\lambda})^{\alpha_1-1}$$

$$f^{(3)}_{X_i|X_j}(x_i|x_j) = \frac{\alpha_3(1-(1-(1-e^{-x_i})^{\gamma})^{\lambda})^{\alpha_1}}{(\alpha_2+\alpha_3)}$$

Proof. The theorem follows readily upon substituting for the joint pdf of (X_1, X_2) in (6), (7) and (8) and the marginal pdf of of $X_i (i = 1, 2)$ in (13), in the relation

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i, x_j)}{f_{X_j}(x_j)}$$

3. Moment Generating Functions

We present the joint moment generating function of (X_1, X_2) and the marginal moment generating function of $X_i (i = 1, 2)$.

3.1 Marginal Moment Generating Functions

We discuss the marginal moment generating function of $X_i (i = 1, 2)$.

Lemma 3.1 If $X_i \sim$ log-EK$(\alpha_i + \alpha_3, \lambda, \gamma)$, then the moment generating function of $X_i (i = 1, 2)$ is given by

$$\begin{aligned} M_{X_i}(t_i) = (\alpha_i+\alpha_3)\lambda\gamma \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\gamma(j+1)-1}{k}\binom{\lambda(s+1)-1}{j} \\ \times \binom{\alpha_i+\alpha_3-1}{s}\frac{(-1)^{s+j+k}}{t_i+1+k} \end{aligned} \tag{14}$$

PROOF. Starting with

$$M_{X_i}(t_i) = E(e^{-t_i x_i}) = \int_{i=0}^{\infty} e^{-t_i x_i} f(x_i) dx_i$$

and substituting for $f_{X_i}(x_i)$ from (13), we get

$$\begin{aligned} M_{X_i}(t_i) = (\alpha_i + \alpha_3)\lambda\gamma \int_{i=0}^{\infty} & e^{-t_i x_i} e^{-x_i} (1 - e^{-x_i})^{\gamma-1} (1 - (1 - e^{-x_i})^{\gamma})^{\lambda-1} \\ & \times (1 - (1 - (1 - e^{-x_i})^{\gamma})^{\lambda})^{\alpha_i+\alpha_3-1} dx_i \end{aligned} \tag{15}$$

using the binomial series expansion we have

$$(1 - (1 - (1 - e^{-x_i})^{\gamma})^{\lambda})^{\alpha_i+\alpha_3-1} = \sum_{s=0}^{\infty} \binom{\alpha_i + \alpha_3 - 1}{s} \times (-1)^s (1 - (1 - e^{-x_i})^{\gamma})^{s\lambda} \tag{16}$$

Substituting from (16) into (15) we get

$$\begin{aligned} M_{X_i}(t_i) = (\alpha_i + \alpha_3)\lambda\gamma \sum_{s=0}^{\infty} & \binom{\alpha_i + \alpha_3 - 1}{s} (-1)^s \int_{i=0}^{\infty} e^{-t_i x_i} \\ & \times e^{-x_i} (1 - e^{-x_i})^{\gamma-1} (1 - (1 - e^{-x_i})^{\gamma})^{\lambda(s+1)-1} dx_i \end{aligned} \tag{17}$$

Substituting from the relation in (16) into (17) we get

$$\begin{aligned} M_{X_i}(t_i) = (\alpha_i + \alpha_3)\lambda\gamma & \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\gamma(j+1) - 1}{k} \binom{\lambda(s+1) - 1}{j} \\ & \times \binom{\alpha_i + \alpha_3 - 1}{s} (-1)^{s+j+k} \int_{i=0}^{\infty} e^{-(t_i+1+k)x_i} dx_i \end{aligned}$$

from which we readily derive the expression of $M_{X_i}(t_i)$ given in (14).

Note that the moment generating function $M_{X_i}(t_i)$ can be used, instead of the marginal pdf $f(x_i)$, to derive the marginal expectation of X_i as

$$E(X_i) = -\frac{d}{dt_i} M_{X_i}(t_i) \bigg|_{t_i=0}$$

From (14), we obtain

$$\begin{aligned} -\frac{d}{dt_i} M_{X_i}(t_i) = (\alpha_i + \alpha_3)\lambda\gamma & \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\gamma(j+1) - 1}{k} \binom{\lambda(s+1) - 1}{j} \\ & \times \binom{\alpha_i + \alpha_3 - 1}{s} \frac{(-1)^{s+j+k}}{(t_i + 1 + k)^2} \end{aligned}$$

in which if we set $t_i = 0$, we obtain $E(X_i)$.

Similarly, the second moment of X_i can be derived from $M_{X_i}(t_i)$ as its second derivative at $t_i = 0$. The expression for the function $M_{X_i}(t_i)$ in (14) can be used to derive the rth moment of X_i as given below

$$E(X_i^r) = -\frac{d^r}{dt_i^r} M_{X_i}(t_i) = (\alpha_i + \alpha_3)\lambda\gamma \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\gamma(j+1)-1}{k} \binom{\lambda(s+1)-1}{j} \times \binom{\alpha_i + \alpha_3 - 1}{s} \frac{r(-1)^{s+j+k}}{(t_i+1+k)^{r+1}}$$

3.2 The Joint Moment Generating Function

The joint moment generating function of (X_1, X_2) can be derived as follows:

Theorem 3.2 If $(X_1, X_2) \sim$ Blog-EK $(\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma)$, then the joint moment generating function of (X_1, X_2) is given by

$$M(t_1,t_2) = \frac{AB}{(t_2+d+1)} - \frac{AB}{(t_1+t_2+k+d+2)} + \frac{A_1B_1}{(t_1+d+1)} - \frac{A_1B_1}{(t_1+t_2+k+d+2)} + \frac{A_2}{(t_1+t_2+k+1)} \tag{18}$$

where

$$A = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\alpha_1+\alpha_3-1}{i}\binom{\lambda(i+1)-1}{j}\binom{\gamma(j+1)-1}{k} \frac{(\alpha_1+\alpha_3)\lambda\gamma(-1)^{i+j+k}}{t_1+k+1}$$

$$B = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty} \binom{\alpha_2-1}{m}\binom{\lambda(m+1)-1}{s}\binom{\gamma(s+1)-1}{d}(-1)^{m+s+d}\alpha_2\lambda\gamma$$

$$A_1 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\alpha_2+\alpha_3-1}{i}\binom{\lambda(i+1)-1}{j}\binom{\gamma(j+1)-1}{k} \frac{(\alpha_2+\alpha_3)\lambda\gamma(-1)^{i+j+k}}{t_2+k+1}$$

$$B_1 = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\sum_{d=0}^{\infty} \binom{\alpha_1-1}{m}\binom{\lambda(m+1)-1}{s}\binom{\gamma(s+1)-1}{d}(-1)^{m+s+d}\alpha_1\lambda\gamma$$

$$A_2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\alpha_1+\alpha_2+\alpha_3-1}{i}\binom{\lambda(i+1)-1}{j}\binom{\gamma(j+1)-1}{k}(-1)^{i+j+k}\alpha_3\lambda\gamma$$

Proof. The joint moment generating function of (X_1, X_2) is given as follows

$$M(t_1,t_2) = E(e^{-(t_1x_1+t_2x_2)}) = \int_0^\infty\int_0^\infty e^{-(t_1x_1+t_2x_2)} f(x_1,x_2)dx_1dx_2$$

$$M(t_1,t_2) = \int_0^\infty\int_0^{x_2} e^{-(t_1x_1+t_2x_2)} f_1(x_1,x_2)dx_1dx_2 + \int_0^\infty\int_0^{x_1} e^{-(t_1x_1+t_2x_2)} f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty e^{-(t_1+t_2)x} f_3(x,x)dx$$

Let

$$J_1 = \int_0^\infty \int_0^{x_2} e^{-(t_1x_1+t_2x_2)} f_1(x_1, x_2)dx_1dx_2$$
$$J_2 = \int_0^\infty \int_0^{x_1} e^{-(t_1x_1+t_2x_2)} f_2(x_1, x_2)dx_2dx_1$$
$$J_3 = \int_0^\infty e^{-(t_1+t_2)x} f_3(x, x)dx$$

Substituting from (6) into J_1 we get

$$J_1 = \int_0^\infty \alpha_2\lambda\gamma e^{-x_2}(1-e^{-x_2})^{\gamma-1}(1-(1-e^{-x_2})^\gamma)^{\lambda-1}(1-(1-(1-e^{-x_2})^\gamma)^\lambda)^{\alpha_2-1}$$
$$\times \int_0^{x_2} (\alpha_1+\alpha_3)\lambda\gamma e^{-x_1}(1-e^{-x_1})^{\gamma-1}(1-(1-e^{-x_1})^\gamma)^{\lambda-1}(1-(1-(1-e^{-x_1})^\gamma)^\lambda)^{\alpha_1+\alpha_3-1}dx_1dx_2$$

using the relation in (16) we obtain

$$J_1 = \frac{AB}{(t_2+d+1)} - \frac{AB}{(t_1+t_2+k+d+2)}$$

Similarly we can obtain J_2 as follows

$$J_2 = \frac{A_1B_1}{(t_1+d+1)} - \frac{A_1B_1}{(t_1+t_2+k+d+2)}$$

And we can obtain J_3 as follows

$$J_3 = \frac{A_2}{(t_1+t_2+k+1)}$$

Then we can obtain the joint moment generating function $M(t_1, t_2) = J_1 + J_2 + J_3$ given in (18).

4. Maximum Likelihood Estimation

Suppose$((x_{11}, x_{12}), \ldots, (x_{1n}, x_{2n}))$ is a random sample from Blog-EK$(\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma)$ distribution. Consider the following notation $n_1 = (i; X_{1i} < X_{2i})$, $n_2 = (i; X_{1i} > X_{2i})$, $n_3 = (i; X_{1i} = X_{2i} = X_i)$ and $n = n_1 + n_2 + n_3$. The likelihood function for the vector of parameters $\theta = (\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma)^\top$ can be expressed as

$$l(\theta) = \prod_{i=1}^{n_1} f_1(x_{1i}, x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i}, x_{2i}) \prod_{i=1}^{n_3} f_3(x_{1i}, x_{2i})$$

Based on the observations, and using the density functions (6), (7) and (8) the likelihood function becomes:

$$
\begin{aligned}
l(\theta) &= ((\alpha_1+\alpha_3)\alpha_2\lambda^2\gamma^2)^{n_1}\prod_{i=1}^{n_1} e^{-x_{1i}}\nu_{1i}(\eta_{1i})^{\lambda-1}(1-(\eta_{1i})^{\lambda})^{\alpha_1+\alpha_3-1}e^{-x_{2i}}\nu_{2i}(\eta_{2i})^{\lambda-1}\\
&\times (1-(\eta_{2i})^{\lambda})^{\alpha_2-1}((\alpha_2+\alpha_3)\alpha_1\lambda^2\gamma^2)^{n_2}\prod_{i=1}^{n_2} e^{-x_{1i}}\nu_{1i}(\eta_{1i})^{\lambda-1}(1-(\eta_{1i})^{\lambda})^{\alpha_1-1}e^{-x_{2i}}\\
&\times \nu_{2i}(\eta_{2i})^{\lambda-1}(1-(\eta_{2i})^{\lambda})^{\alpha_2+\alpha_3-1}(\alpha_3\lambda\gamma)^{n_3}\prod_{i=1}^{n_3} e^{-x_i}\nu_i(\eta_i)^{\lambda-1}(1-(\eta_i)^{\lambda})^{\alpha_1+\alpha_2+\alpha_3-1}
\end{aligned}
$$

The log-likelihood function can be written as

$$
\begin{aligned}
L(\theta) &= n_1\ln((\alpha_1+\alpha_3)\alpha_2\lambda^2\gamma^2) - \sum_{i=1}^{n_1}x_{1i} + \sum_{i=1}^{n_1}ln(\nu_{1i}) + (\lambda-1)\sum_{i=1}^{n_1}ln(\eta_{1i})\\
&+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}ln(1-(\eta_{1i})^{\lambda}) - \sum_{i=1}^{n_1}x_{2i} + \sum_{i=1}^{n_1}ln(\nu_{2i}) + (\lambda-1)\sum_{i=1}^{n_1}ln(\eta_{2i})\\
&+(\alpha_2-1)\sum_{i=1}^{n_1}ln(1-(\eta_{2i})^{\lambda}) + n_2ln((\alpha_2+\alpha_3)\alpha_1\lambda^2\gamma^2) - \sum_{i=1}^{n_2}x_{1i} + \sum_{i=1}^{n_2}ln(\nu_{1i})\\
&+(\lambda-1)\sum_{i=1}^{n_2}ln(\eta_{1i}) + (\alpha_1-1)\sum_{i=1}^{n_2}ln(1-(\eta_{1i})^{\lambda}) - \sum_{i=1}^{n_2}x_{2i} + \sum_{i=1}^{n_2}ln(\nu_{2i})\\
&+(\lambda-1)\sum_{i=1}^{n_2}ln(\eta_{2i}) + (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}ln(1-(\eta_{2i})^{\lambda}) + n_3ln(\alpha_3\lambda\gamma) - \sum_{i=1}^{n_3}x_i\\
&+\sum_{i=1}^{n_3}ln(\nu_i) + (\lambda-1)\sum_{i=1}^{n_3}ln(\eta_i) + (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}ln(1-(\eta_i)^{\lambda})
\end{aligned} \tag{19}
$$

Computing the first partial derivatives of (19) with respect to $\alpha_1, \alpha_2, \alpha_3$ and λ, and setting the results equal zeros, we get the likelihood equations as in the following form

$$
\frac{\partial L(\theta)}{\partial\alpha_1} = \frac{n_1}{(\alpha_1+\alpha_3)} + \frac{n_2}{\alpha_1} + \sum_{i=1}^{n_1}ln(1-(\eta_{1i})^{\lambda}) + \sum_{i=1}^{n_2}ln(1-(\eta_{1i})^{\lambda}) + \sum_{i=1}^{n_3}ln(1-(\eta_i)^{\lambda}) \tag{20}
$$

$$
\frac{\partial L(\theta)}{\partial\alpha_2} = \frac{n_1}{\alpha_2} + \frac{n_2}{(\alpha_2+\alpha_3)} + \sum_{i=1}^{n_1}ln(1-(\eta_{2i})^{\lambda}) + \sum_{i=1}^{n_2}ln(1-(\eta_{2i})^{\lambda}) + \sum_{i=1}^{n_3}ln(1-(\eta_i)^{\lambda}) \tag{21}
$$

$$
\begin{aligned}
\frac{\partial L(\theta)}{\partial\alpha_3} &= \frac{n_1}{(\alpha_1+\alpha_3)} + \frac{n_2}{(\alpha_2+\alpha_3)} + \frac{n_3}{\alpha_3} + \sum_{i=1}^{n_1}ln(1-(\eta_{1i})^{\lambda}) + \sum_{i=1}^{n_2}ln(1-(\eta_{2i})^{\lambda})\\
&+\sum_{i=1}^{n_3}ln(1-(\eta_i)^{\lambda})
\end{aligned} \tag{22}
$$

$$\begin{aligned}\frac{\partial L(\theta)}{\partial \lambda} &= \frac{2n_1}{\lambda} + \sum_{i=1}^{n_1} ln(\eta_{1i}) - (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\frac{(\eta_{1i})^\lambda ln(\eta_{1i})}{(1-(\eta_{1i})^\lambda)} + \sum_{i=1}^{n_1} ln(\eta_{2i})\\ &\quad - (\alpha_2-1)\sum_{i=1}^{n_1}\frac{(\eta_{2i})^\lambda ln(\eta_{2i})}{(1-(\eta_{2i})^\lambda)} + \frac{2n_2}{\lambda} + \sum_{i=1}^{n_2} ln(\eta_{1i}) - (\alpha_1-1)\sum_{i=1}^{n_2}\frac{(\eta_{1i})^\lambda ln(\eta_{1i})}{(1-(\eta_{1i})^\lambda)}\\ &\quad + \sum_{i=1}^{n_2} ln(\eta_{2i}) - (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}\frac{(\eta_{2i})^\lambda ln(\eta_{2i})}{(1-(\eta_{2i})^\lambda)} + \frac{n_3}{\lambda} + \sum_{i=1}^{n_3} ln(\eta_i)\\ &\quad - (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}\frac{(\eta_i)^\lambda ln(\eta_i)}{(1-(\eta_i)^\lambda)}\end{aligned} \tag{23}$$

To get the MLEs of the parameters $\alpha_1, \alpha_2, \alpha_3$ and λ, we have to solve the above system of four non-linear equations with respect to $\alpha_1, \alpha_2, \alpha_3$ and λ. The solution of equations (20), (21), (22) and (23) is not possible in closed form, so numerical technique is needed to get the MLEs.

The approximate confidence intervals of the parameters based on the asymptotic distributions of their MLEs are derived. For the observed information matrix of $\alpha_1, \alpha_2, \alpha_3$ and λ, we find the second partial derivatives as follows

$$\frac{\partial^2 L(\theta)}{\partial \alpha_1^2} = I_{11} = -\frac{n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{\alpha_1^2} \quad \frac{\partial^2 L(\theta)}{\partial \alpha_1 \partial \alpha_2} = I_{12} = 0 \quad \frac{\partial^2 L(\theta)}{\partial \alpha_1 \partial \alpha_3} = I_{13} = -\frac{n_1}{(\alpha_1+\alpha_3)^2}$$

$$\frac{\partial^2 L(\theta)}{\partial \alpha_1 \partial \lambda} = I_{14} = -\sum_{i=1}^{n_1}\frac{(\eta_{1i})^\lambda ln(\eta_{1i})}{(1-(\eta_{1i})^\lambda)} - \sum_{i=1}^{n_2}\frac{(\eta_{1i})^\lambda ln(\eta_{1i})}{(1-(\eta_{1i})^\lambda)} - \sum_{i=1}^{n_3}\frac{(\eta_i)^\lambda ln(\eta_i)}{(1-(\eta_i)^\lambda)}$$

$$\frac{\partial^2 L(\theta)}{\partial \alpha_2^2} = I_{22} = -\frac{n_1}{(\alpha_2)^2} - \frac{n_2}{(\alpha_2+\alpha_3)^2}$$

$$\frac{\partial^2 L(\theta)}{\partial \alpha_2 \partial \alpha_3} = I_{23} = -\frac{n_2}{(\alpha_2+\alpha_3)^2}$$

$$\frac{\partial^2 L(\theta)}{\partial \alpha_2 \partial \lambda} = I_{24} = -\sum_{i=1}^{n_1}\frac{(\eta_{2i})^\lambda ln(\eta_{2i})}{(1-(\eta_{2i})^\lambda)} - \sum_{i=1}^{n_2}\frac{(\eta_{2i})^\lambda ln(\eta_{2i})}{(1-(\eta_{2i})^\lambda)} - \sum_{i=1}^{n_3}\frac{(\eta_i)^\lambda ln(\eta_i)}{(1-(\eta_i)^\lambda)}$$

$$\frac{\partial^2 L(\theta)}{\partial \alpha_3^2} = I_{33} = -\frac{n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{(\alpha_2+\alpha_3)^2} - \frac{n_3}{(\alpha_3)^2}$$

$$\frac{\partial^2 L(\theta)}{\partial \alpha_3 \partial \lambda} = I_{34} = -\sum_{i=1}^{n_1}\frac{(\eta_{1i})^\lambda ln(\eta_{1i})}{(1-(\eta_{1i})^\lambda)} - \sum_{i=1}^{n_2}\frac{(\eta_{2i})^\lambda ln(\eta_{2i})}{(1-(\eta_{2i})^\lambda)} - \sum_{i=1}^{n_3}\frac{(\eta_i)^\lambda ln(\eta_i)}{(1-(\eta_i)^\lambda)}$$

$$\begin{aligned}\frac{\partial^2 L(\theta)}{\partial \lambda^2} = I_{44} &= -\frac{2n_1}{\lambda^2} - (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\frac{(\eta_{1i})^\lambda (ln(\eta_{1i}))^2}{(1-(\eta_{1i})^\lambda)^2} - (\alpha_2-1)\sum_{i=1}^{n_1}\frac{(\eta_{2i})^\lambda (ln(\eta_{2i}))^2}{(1-(\eta_{2i})^\lambda)^2}\\ &\quad - \frac{2n_2}{\lambda^2} - (\alpha_1-1)\sum_{i=1}^{n_2}\frac{(\eta_{1i})^\lambda (ln(\eta_{1i}))^2}{(1-(\eta_{1i})^\lambda)^2} - (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}\frac{(\eta_{2i})^\lambda (ln(\eta_{2i}))^2}{(1-(\eta_{2i})^\lambda)^2} - \frac{n_3}{\lambda^2}\\ &\quad - (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}\frac{(\eta_i)^\lambda (ln(\eta_i))^2}{(1-(\eta_i)^\lambda)^2}\end{aligned}$$

where $\nu_1 = (1-e^{-x_1})^{\gamma-1}$, $\eta_1 = 1-(1-e^{-x_1})^\gamma$, $\nu_2 = (1-e^{-x_2})^{\gamma-1}$, $\eta_2 = 1-(1-e^{-x_2})^\gamma$, $\nu = (1-e^{-x})^{\gamma-1}$, $\eta = 1-(1-e^{-x_1})^\gamma$.

Then the observed information matrix is given by

$$\mathbf{I} = -\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}$$

so the variance-covariance matrix may be approximated as

$$\mathbf{V} = -\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}^{-1} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix}$$

5. Simulation and Data Analysis

In this section first we present Monte Carlo simulation results to study the behavior of the MLEs and then present one data analysis results mainly for illustrative purpose.

5.1 Simulation Results

In this subsection we present some simulation results to see how the MLEs behave for different sample sizes and for different initial parameter values. We have used different sample sizes namely n = 20, 40, 60, 80 and 100 and two different sets of parameter values: Set 1: $\alpha_1 = 1.1, \alpha_2 = 1.2, \alpha_3 = 1.3, \lambda = 1$ and Set 2: $\alpha_1 = 1.12, \alpha_2 = 1.23, \alpha_3 = 1.34, \lambda = 1$. In each case we have computed the MLEs of the unknown parameters by maximizing the log-likelihood function (19). We compute the average estimates and mean square error over 1000 replications and the results are reported in Table 1.

Table 1. The average of MLEs and the associated mean square errors (within brackets below).

	Set:1				Set:2			
n	α_1	α_2	α_3	λ	α_1	α_2	α_3	λ
$n = 20$	1.321	1.435	1.455	1.087	1.319	1.439	1.549	1.069
	(0.536)	(0.682)	(0.735)	(0.055)	(0.508)	(0.638)	(0.757)	(0.051)
$n = 40$	1.199	1.298	1.392	1.036	1.225	1.333	1.441	1.037
	(0.183)	(0.246)	(0.294)	(0.021)	(0.220)	(0.298)	(0.346)	(0.023)
$n = 60$	1.167	1.282	1.358	1.029	1.191	1.292	1.426	1.026
	(0.119)	(0.182)	(0.222)	(0.014)	(0.134)	(0.193)	(0.281)	(0.013)
$n = 80$	1.146	1.255	1.345	1.019	1.168	1.288	1.386	1.019
	(0.091)	(0.147)	(0.186)	(0.011)	(0.106)	(0.163)	(0.215)	(0.010)
$n = 100$	1.137	1.252	1.338	1.017	1.163	1.283	1.380	1.017
	(0.068)	(0.124)	(0.167)	(0.008)	(0.079)	(0.144)	(0.198)	(0.008)

Some of the points are quite clear from Table 1. In all the cases the performances of the maximum likelihood estimate are quite satisfactory. It is observed that as sample size increases the average estimates and the mean squared error decrease for all the parameters, as expected.

5.2 Data Analysis

The following data represent the American Football (National Football League) League data and they are obtained from the matches played on three consecutive weekends in 1986. The data were first published in 'Washington Post' and they are also available in Csorgo and Welsh (1989).

It is a bivariate data set, and the variables X_1 and X_2 are as follows: X_1 represents the 'game time' to the first points scored by kicking the ball between goal posts, and X_2 represents the 'game time' to the first points scored by moving the ball into the end zone. These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game.

The data (scoring times in minutes and seconds) are represented in Table 2. The data set was first analyzed by Csorgo and Welsh (1989), by converting the seconds to the decimal minutes, i.e. 2:03 has been converted to 2.05, 3:59 to 3.98 and so on. We have also adopted the same procedure. Here also all the data points are divided by 10 just for computational purposes. It should not make any difference in the statistical inference. We have taken the initial guesses of $\alpha_1, \alpha_2, \alpha_3$ and λ are all equal to 1.

Table 2. American Football League (N F L) data

X_1	X_2	X_1	X_2	X_1	X_2
2.05	3.98	5.78	25.98	10.40	10.25
9.05	9.05	13.80	49.75	2.98	2.98
0.85	0.85	7.25	7.25	3.88	6.43
3.43	3.43	4.25	4.25	0.75	0.75
7.78	7.78	1.65	1.65	11.63	17.37
10.57	14.28	6.42	15.08	1.38	1.38
7.05	7.05	4.22	9.48	10.53	10.53
2.58	2.58	15.53	15.53	12.13	12.13
7.23	9.68	2.90	2.90	14.58	14.58
6.85	34.58	7.02	7.02	11.82	11.82
32.45	42.35	6.42	6.42	5.52	11.27
8.53	14.57	8.98	8.98	19.65	10.71
13.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	8.87	8.87	10.85	38.07

The variables X_1 and X_2 have the following structure: (i) $X_1 < X_2$ means that the first score is a field goal, (ii) $X_1 = X_2$ means the first score is a converted touchdown, (iii) $X_1 > X_2$ means the first score is an unconverted touchdown or safety. In this case the ties are exact because no 'game time' elapses between a touchdown and a point-after conversion attempt. Therefore, here ties occur quite naturally and they can not be ignored. It should be noted that the possible scoring times are restricted by the duration of the game but it has been ignored similarly as in Csorgo and Welsh (1989).

If we define the following random variables:

U_1 = time to first field goal
U_2= time to first safety or unconverted touchdown
U_3 = time to first converted touchdown,

then $X_1 = max(U_1, U_3)$, $X_2 = max(U_2, U_3)$. Therefore, (X_1, X_2) has a similar structure as the Marshall-Olkin bivariate exponential model or the proposed Blog-EK model.

Before going to analyze the data using Blog-EK model, we fit the log-EK distribution to X_1 and X_2 separately. The MLEs of α and λ of the respective log-EK distribution for X_1 and X_2 are (1.848, 1.582) and (1.396, 0.917) respectively. The Kolmogorov-Smirnov distances between the fitted distribution and the empirical distribution function and the corresponding p values (in brackets) for X_1 and X_2 are 0.0868 (0.5628) and 0.0867 (0.5617) respectively. Based on the p values log-EK distribution can not be rejected for the marginals.

The MLEs of $\alpha_1, \alpha_2, \alpha_3$ and λ are obtained by maximizing the log-likelihood function (19) with respect to the four unknown parameters and they are as follows 0.048, 0.596, 1.171 and 0.956 respectively. The corresponding log-likelihood value is -100.139. The corresponding 95% confidence intervals are (0.046, 0.142), (0.676, 0.92), (0.698, 1.643), (0.647, 1.266) respectively.

Now the natural question is how good the fit. Unfortunately, we do not have any proper bivariate goodness of fit test for general models like the univariate case. We examine the marginals and the maximum of the marginals, definitely they provide some indication about the goodness of fit of the proposed Blog-EK model to the given data set. We fit log-EK (1.925, 1.619), log-EK (1.393, 0.916) and log-EK (1.384, 0.898) to X_1, X_2 and X_3 respectively. The parameters of the corresponding log-EK model are obtained from Corollary 2.6, by replacing the true values with their estimates.

The Kolmogorov-Smirnov distances between the empirical distribution function and the fitted distribution function and the associated p values reported in brackets in three cases are 0.0827 (0.5357), 0.0867 (0.5617) and 0. 0.0703 (0.4556) respectively. From the p values, we cannot reject the hypotheses that X_1, X_2 and X_3 follow log-EK. We have fitted four-parameter Blog-EK model also to this data set.

We analyze the data using the Blog-EK model. We have taken the initial guesses of $\alpha_1, \alpha_2, \alpha_3$ and λ are all equal to 1. The estimate of $\alpha_1, \alpha_2, \alpha_3$ and λ become 0.048, 0.596, 1.171 and 9.563 respectively. The corresponding log-likelihood value is 38.017. the 95% confidence intervals of $\alpha_1, \alpha_2, \alpha_3$ and λ are (0, 0.142), (0.309, 0.883), (0.793, 1.548), (8.531, 10.596) respectively.

6. Conclusions

In this paper we have proposed Blog-EK distribution function whose marginals are log-EK distributions. This new bivariate distribution has several interesting properties and it can be used as an alternative to the several continuous bivariate distributions. The generation of random samples from proposed bivariate distribution is very simple, and therefore Monte Carlo simulation can be performed very easily for different statistical inference purpose. It is observed that the MLEs of the unknown parameters can be obtained by solving four non-linear equations and Monte Carlo simulation indicate that the performance of the MLEs are quite satisfactory. Analysis of one real data indicates that the performance of the confidence intervals based on asymptotic distribution.

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