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Comparative study of Bhattacharyya and Kshirsagar bounds in Burr XII and Burr III distributions

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Abstract

A set of families of distributions which might be useful for fitting data was described by Burr (1942). Among them, the families type XII (Burr XII) and type III (Burr III), have gathered special attention in physics, actuarial studies, reliability and applied statistics. Estimating a wide range of functions of their parameters such as reliability, hazard rate and mode, under various conditions, have been done. But, the variances of the estimators are not considered precisely yet.

In this paper, we consider two well-known lower bounds for the variance of any unbiased estimator, which are Bhattacharyya and Kshirsagar bounds for the Burr XII and Burr III distributions. In these distributions, the general forms of the Bhattacharyya and Kshirsagar matrices are obtained. In addition, we evaluate different Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the reliability, hazard rate, mode and median due to Burr XII and Burr III distributions and conclude that in each case, which bound has higher convergence and is better to use. Also via some figures, we compare the two bounds with bootstrap method in approximating the variance of the unbiased estimator of the reliability, median and mean of the Burr XII distributions.

Keywords: Bhattacharyya bound · Bootstrap method · Cramer-Rao bound · Hammersley-Chapman-Robins bound · hazard rate · Kshirsagar bound · reliability function.

Mathematics Subject Classification: 62P30 · 62F99 · 65D20.

1. Introduction

Burr (1942) introduced 12 families of distributions based on the differential equation $\frac{dF(x)}{dx} = F(x)(1 - F(x))g(x; F(x))$. Of these, Burr XII and Burr III distributions are two of the most versatile distributions in statistics especially in reliability aspects.

The Burr XII is a unimodal distribution and has a non-monotone hazard function, which can accommodate many shapes of it. Thus, the use of this distribution as a failure model is appropriate and useful in applied statistics, especially in survival analysis and actuarial studies.

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As shown by many authors like, Burr and Cislak (1968), Burr (1968), Rodriguez (1977) and Tadikamalla (1980), if one chooses the parameters appropriately, the Burr XII distribution contains the shape characteristics of the normal, log-normal, gamma, logistic and exponential (Pearson type X) distributions, as well as a significant portion of the Pearson types I (beta), II, III (gamma), V, VII, IX and XII families. Other particular cases of the Burr XII, include Fisher (F), inverted beta, Lomax, Pareto and the log-logistic distributions. It is therefore observable that the versatility and flexibility of the Burr XII distribution make it quite attractive as a tentative and empirical model for data whose underlying distribution is unknown.

Wingo (1983, 1993) has described methods for fitting the Burr XII distribution to life test or other (complete sample) data by maximum likelihood and has also provided an extensive list of references to earlier published work on this distribution. Other researchers who have studied the usefulness and properties of the Burr XII distribution include Papadopoulos (1978), Evans and Ragab (1983), Al-Hussaini and Jaheen (1992), Wang et al. (1996), Al-Yousef (2002), Soliman (2002, 2005), Wang et al. (2007) and Ahmad et al. (2011).

A lower bound for the variance of an estimator is one of the fundamental things in the estimation theory because it gives us an idea about the accuracy of an estimator. When the variance has complicated form and we can not compute it, by lower bounds, we can approximate it. Up to now, many studies have been done for the lower bound of the variance of an unbiased estimator of the parameter. The well-known lower bounds are Cramer-Rao, Bhattacharyya, Hammersley-Chapman-Robins, Kshirsagar and Koike.

In this paper, according to usefulness and wide applications of the Burr XII and Burr III distributions and importance of finding and approximating a lower bound for the variance of the estimators, we first introduce the most sharper bounds which are the Bhattacharyya bound under regularity conditions and Kshirsagar bound under non-regularity conditions. Then, we construct the general forms of the Bhattacharyya and Kshirsagar matrices which are used in their inequalities. Also, we evaluate and compare different Bhattacharyya and Kshirsagar bounds for the variance of estimator of some applicable functions such as the reliability function, hazard rate, mode, median and mean in Burr XII and Burr III distributions. Furthermore, some graphical comparisons among two lower bounds and bootstrap method (Efron, 1979) have been done.

2. Bhattacharyya bound

Bhattacharyya (1946, 1947) obtained a generalized form of the Cramer-Rao inequality which is related to the Bhattacharyya matrix. The Bhattacharyya matrix is the covariance matrix of the random vector,

$$\frac{1}{f(X|\theta)}(f^{(1)}(X|\theta), f^{(2)}(X|\theta), \ldots, f^{(k)}(X|\theta)),$$

where $f^{(j)}(.|\theta)$ is the j^{th} derivative of the probability density function $f(.|\theta)$ w.r.t. the parameter θ. The covariance matrix of the above random vector is referred to as the $k \times k$ Bhattacharyya matrix and k is the order of it. It is clear that $(1,1)^{th}$ element of the Bhattacharyya matrix is the Fisher information.

Under some regularity conditions, the Bhattacharyya bound for any unbiased estimator of the $g(\theta)$ is defined as follows,

$$Var_\theta(T(X)) \geq \mathbf{J}_\theta \mathbf{W}^{-1} \mathbf{J}_\theta^t := B_k(\theta), \tag{1}$$

where t refers to the transpose, $\mathbf{J}_\theta = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(k)}(\theta))$, $g^{(j)}(\theta) = \partial^j g(\theta)/\partial\theta^j$ for

$j = 1, 2, \ldots, k$ and $\mathbf{W}^{-1}$ is the inverse of the Bhattacharyya matrix, where

$$\mathbf{W} = (W_{rs}) = \left(Cov_\theta \left\{ \frac{f^{(r)}(X|\theta)}{f(X|\theta)}, \frac{f^{(s)}(X|\theta)}{f(X|\theta)} \right\} \right),$$

such that $E_\theta(\frac{f^{(r)}(X|\theta)}{f(X|\theta)}) = 0$ for $r, s = 1, 2, \ldots, k$.

If we substitute $k = 1$ in (1), then it indeed reduces to the Cramer-Rao inequality. By using the properties of the multiple correlation coefficient, it is easy to show that as the order of the Bhattacharyya matrix (k) increases, the Bhattacharyya bound becomes sharper.

Shanbhag (1972, 1979) characterized the natural exponential family with quadratic variance function (NEF-QVF) via diagonality of the Bhattacharyya matrix, and also showed that for this family, the Bhattacharyya matrix of any order exists and is diagonal. One can see more details and information about Bhattacharyya bound in the papers such as, Blight and Rao (1974), Tanaka and Akahira (2003), Tanaka (2003, 2006), Mohtashami Borzadaran (2006), Khorashadizadeh and Mohtashami (2007), Mohtashami Borzadaran et al. (2010).

3. Kshirsagar bound

It is well-known that, the Hammersley-Chapman-Robbins is a sharper lower bound than Cramer-Rao which needs no regularity conditions. This lower bound has been introduced independently by Hammersley (1950) and Chapman and Robbins (1951).

If there exists ϕ, such that $\phi \in \Theta$ and $S(\phi) \subset S(\theta)$, where $S(\theta) = \{x | f(x|\theta) > 0\}$, then the Hammersley-Chapman-Robbins lower bound says that,

$$Var_\theta(T(X)) \geq \sup_\phi \frac{[g(\phi) - g(\theta)]^2}{E_\theta \left(\frac{f(X|\phi) - f(X|\theta)}{f(X|\theta)} \right)^2}. \tag{2}$$

Sen and Ghosh (1976) gave the conditions for the attainment of the inequality and also they compared this bound with Bhattacharyya bound and provided the sufficient conditions to determine when one bound is sharper than the other.

Qin and Nayak (2008) obtained the Kshirsagar lower bounds for mean squared error of prediction. Akahira and Ohyauchi (2007) considered a Bayesian view of the Hammersley-Chapman-Robbins inequality.

Furthermore, the bound is also derived directly by Akahira and Ohyauchi (2003) and Ohyauchi (2004), using the Lagrange method.

Recently, Kshirsagar (2000) extended the Hammersley-Chapman-Robbins lower bound in the same manner of the Bhattacharyya inequality. This bound does not need the assumptions of the common support and the existence of the derivative of the density function. The Kshirsagar bound states that for any unbiased estimator $T(X)$ of $g(\theta)$,

$$Var_\theta(T(X)) \geq \sup_\phi \lambda_\theta^t \Sigma^{-1} \lambda_\theta := K_k(\theta), \tag{3}$$

where t refers to the transpose, $\lambda_\theta = (g(\phi_1) - g(\theta), g(\phi_2) - g(\theta), \ldots, g(\phi_k) - g(\theta))^t$ and Σ^{-1} is the inverse of matrix with elements as follow,

$$\Sigma_{rs} = Cov_\theta(\psi_r, \psi_s), \quad r, s = 1, 2, \ldots, k,$$

where, $\psi_r = \frac{f(X|\phi_r) - f(X|\theta)}{f(X|\theta)}$ and the supremum is taken over the set of all $\phi_i \in \Theta, (i = 1, 2, \ldots, k)$, satisfying,

$$S(\phi_k) \subset S(\phi_{k-1}) \subset \ldots \subset S(\phi_1) \subset S(\theta).$$

Kshirsagar (2000) showed that for fixed k, this bound is sharper than the Bhattacharyya bound order k. Although, computing the Kshirsagar bound and taking the supremums are difficult, but, nowadays, using computers make it a little easier to compute.

Koike (2002) considered another extension of Hammersley-Chapman-Robbins bound in the same manner as Kshirsagar and some relations with usual Bhattacharyya bound. Koike (2002) showed that, his proposed bound is sharper than Bhattacharyya bound and weaker than Kshirsagar bound and by choosing $\phi_i = \theta + i\delta$ in (3), Kshirsagar bound and his bound are equal. Nayeban et al. (2013, 2014) have been compared Kshirsagar and Bhattacharyya bounds in different family of distributions.

4. Bhattacharyya and Kshirsagar bounds in Burr XII and Burr III distributions

Let X and Y have Burr XII and Burr III distributions respectively with probability density function (pdf) as,

$$f(x) = \frac{\alpha\theta x^{\alpha-1}}{(1 + x^\alpha)^{\theta+1}}; \quad x > 0, \alpha > 0, \theta > 0, \tag{4}$$

$$f(y) = \frac{\alpha\theta y^{-\alpha-1}}{(1 + y^{-\alpha})^{\theta+1}}; \quad y > 0, \alpha > 0, \theta > 0. \tag{5}$$

Also, their corresponding cumulative distribution functions (cdf) are respectively as follows,

$$F(x) = 1 - (1 + x^\alpha)^{-\theta},$$
$$F(y) = (1 + y^{-\alpha})^{-\theta},$$

where α and θ are the shape parameters. It is easily seen that the Burr III is the simple transformation, $Y = \frac{1}{X}$, of Burr XII and therefore it retains most of the properties of (4). The rth moment corresponding (4) and (5) can be written respectively, as,

$$E(X^r) = \theta \frac{\Gamma(\theta - \frac{r}{\alpha})\Gamma(1 + \frac{r}{\alpha})}{\Gamma(\theta + 1)}; \quad \alpha\theta > r,$$
$$E(Y^r) = \theta \frac{\Gamma(\theta + \frac{r}{\alpha})\Gamma(1 - \frac{r}{\alpha})}{\Gamma(\theta + 1)}; \quad \alpha > r.$$

The Burr XII distribution has been used in quality control and reliability by many authors such as, Cook and Johnson (1986), Yourstone and Zimmer (1992), Zimmer et al. (1998), Soliman (2002, 2005) and Asgharzadeh and Valiollahi (2008). Zimmer et al. (1998)

and Wang et al. (2003) discussed the statistical and probabilistic properties of the Burr XII distribution, and its relationship to other distributions used in reliability analysis.

Here, the thing, that is very important, is the variances of the estimators. In what follows, we try to evaluate the some sharp bounds for the variance of all unbiased estimators of $g(\theta)$ in Burr XII and Burr III distributions.

We see that for the matrices of order more than 5, the differences of the bounds are about less than 0.0001, so, we calculate the 5×5 Bhattacharyya and Kshirsagar matrices.

In this paper, we consider θ as unknown parameter as an example. Similar results can be obtained when α is unknown or furthermore in multiparameter case when both parameters are unknown. But the multiparameter version of the Kshirsagar bound has not been study yet and is the future of this work.

By some mathematical computation, it is easy to see that the term $\frac{f^{(r)}(X|\theta)}{f(X|\theta)}$ in Burr XII and Bur III are as follows respectively,

$$\frac{f^{(r)}(X|\theta)}{f(X|\theta)} = \begin{cases} \frac{1-\ln(1+X^{\alpha})^{\theta}}{\theta}; & r = 1 \\ \frac{(-1)^r}{\theta}[\ln(1+X^{\alpha})^{\theta} - r]\ln(1+X^{\alpha})^{r-1}; & r = 2,3,\ldots \end{cases}$$

$$\frac{f^{(r)}(Y|\theta)}{f(Y|\theta)} = \begin{cases} \frac{1-\ln(1+Y^{-\alpha})^{\theta}}{\theta}; & r = 1 \\ \frac{(-1)^r}{\theta}[\ln(1+Y^{-\alpha})^{\theta} - r]\ln(1+Y^{-\alpha})^{r-1}; & r = 2,3,\ldots \end{cases}$$

So, using above equations, we obtained the general form of the 5×5 Bhattacharyya matrix in both Burr XII and Bur III, as follows,

$$\mathbf{W} = \begin{pmatrix} \frac{1}{\theta^2} & \frac{-2}{\theta^3} & \frac{6}{\theta^4} & \frac{-24}{\theta^5} & \frac{120}{\theta^6} \\ & \frac{8}{\theta^4} & \frac{-36}{\theta^5} & \frac{192}{\theta^6} & \frac{-1200}{\theta^7} \\ & & \frac{216}{\theta^6} & \frac{-1440}{\theta^7} & \frac{10800}{\theta^8} \\ & & & \frac{11520}{\theta^8} & \frac{-100800}{\theta^9} \\ & & & & \frac{1008000}{\theta^{10}} \end{pmatrix}. \tag{6}$$

As an example for W_{11}, the $(1,1)^{th}$ element of the matrix, we have,

$$\begin{aligned} W_{11} &= E\left(\frac{f^{(1)}(X|\theta)}{f(X|\theta)}.\frac{f^{(1)}(X|\theta)}{f(X|\theta)}\right) \\ &= \int_0^{\infty} \frac{\alpha x^{\alpha-1}}{\theta(1+x^{\alpha})^{\theta+1}}(\ln(1+x^{\alpha})^{\theta} - 1)^2 dx \\ &= -\frac{[1+x^{\alpha}][1+\theta^2\ln(1+x^{\alpha})^2]}{\theta^2(1+x^{\alpha})^{\theta+1}}|_0^{\infty} \\ &= \frac{1}{\theta^2}, \end{aligned}$$

or for W_{24} ,

$$
\begin{aligned}
W_{24} &= E\left(\frac{f^{(2)}(X|\theta)}{f(X|\theta)}.\frac{f^{(4)}(X|\theta)}{f(X|\theta)}\right) \\
&= \int_0^\infty \frac{\alpha x^{\alpha-1}\ln(1+x^\alpha)^4(\ln(1+x^\alpha)^\theta - 2)(\ln(1+x^\alpha)^\theta - 4)}{\theta(1+x^\alpha)^{\theta+1}}dx \\
&= \frac{192}{\theta^6}.
\end{aligned}
$$

Also, in Kshirsagar bound, by supposing $\phi_i = \theta + i\delta$ for $i = 1, 2, \ldots, k$, where $\delta > -\frac{\theta}{i}$, one can see that in Burr XII the elements of the Kshirsagar matrix are given by,

$$
\begin{aligned}
\Sigma_{rs} = E_\theta(\psi_r.\psi_s) &= \int_0^\infty \frac{[f(X|\phi_r) - f(x|\theta)][f(X|\phi_s) - f(X|\theta)]}{f(X|\theta)}dx \\
&= \int_0^\infty \frac{f(X|\phi_r)f(X|\phi_s)}{f(X|\theta)}dx - 1 \\
&= \int_0^\infty \frac{1}{\theta}\alpha x^{\alpha-1}(\theta + r\delta)(\theta + s\delta)(1+x^\alpha)^{-\delta(r+s)-\theta-1}dx - 1 \\
&= \frac{rs\delta^2}{\theta[(r+s)\delta + \theta]}; \quad r, s = 1, 2, \ldots, k,
\end{aligned}
$$

and in similar way for Burr III we obtain,

$$
\Sigma_{rs} = E_\theta(\psi_r.\psi_s) = -\frac{rs\delta^2}{\theta[(r+s)\delta + \theta]}; \quad r, s = 1, 2, \ldots, k. \tag{7}
$$

In the next subsections we evaluate and compare the Bhattacharyya and Kshirsagar bounds for some applicable parameter functions. Also, some comparison have been done with bootstrap method.

4.1 Lower bounds for the variance of any unbiased estimator of the reliability function in Burr XII and Burr III

The reliability functions of the Burr XII and Burr III distributions are respectively, as follows,

$$
\begin{aligned}
R(x) &= (1+x^\alpha)^{-\theta}, \quad x > 0, \alpha > 0, \theta > 0, \\
R(y) &= 1 - (1+y^{-\alpha})^{-\theta} \quad y > 0, \alpha > 0, \theta > 0.
\end{aligned}
$$

Estimation of the reliability function of some equipments is one of the main problems of reliability theory. In most practical applications and life-test experiments, the distributions with positive domain, e.g.,Weibull, Burr XII, Burr III, Pareto, beta, and Rayleigh, are quite appropriate models. There have been many papers on estimating the reliability function of these distributions in non-Bayes as well as Bayes contexts, e.g., Zacks (1992), Sun and Berger (1994), Meeker and Escobar (1998) and Pensky and Singh (1999).

Specifically, for the Burr XII, Evans and Ragab (1983) obtained Bayes estimates of θ and the reliability function based on type II censored samples. Ali Mousa and Jaheen (2002) obtained Bayes estimation of the two parameters and the reliability function of Burr

XII distribution based on progressive type II censored samples. Also, Based on complete samples, Moore and Papadopoulos (2000) obtained Bayes estimates of θ and the reliability function when the parameter α is assumed to be known.

So, here, we want to approximate the variance of the unbiased estimator of the parameter functions $g(\theta) = (1+a)^{-\theta}$ in Burr XII and $g(\theta) = 1-(1+b)^{-\theta}$ in Burr III, (where a and b are positive and constant) using Bhattacharyya and Kshirsagar bounds. In the Tables 1 and 2, $B_1, \ldots, B_5$ and $K_1, \ldots, K_5$ represents the first five Bhattacharyya and Kshirsagar bounds respectively, for different values of θ, a and b.

Table 1. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the reliability function in Burr XII

θ	a	B_1	B_2	B_3	B_4	B_5
0.1	0.2	0.00032	0.00063	0.00094	0.00124	0.00154
0.5	2	0.10057	0.15349	0.17874	0.18903	0.19216
1	4	0.10361	0.10756	0.11083	0.11971	0.12712
2	6	0.00630	0.01195	0.01280	0.01329	0.01457
3	2	0.01490	0.02115	0.02466	0.02466	0.02589
θ	a	K_1	K_2	K_3	K_4	K_5
0.1	0.2	0.014420	0.014655	0.014734	0.014771	0.016460
0.5	2	0.187019	0.207298	0.211920	0.22005	0.224273
1	4	0.106320	0.125115	0.132446	0.136643	0.140774
2	6	0.009103	0.012488	0.014013	0.014851	0.015383
3	2	0.018009	0.023722	0.026178	0.027357	0.028808

We see that, as the order of Bhattacharyya and Kshirsagar matrices increase, the bounds get bigger and nearer to the exact value of the variance. Here, the important point is that, although evaluating the Kshirsagar bounds are difficult because of taking supremums, but, they are more sharper than their corresponding Bhattacharyya bounds.

In Figure 1 the Bhattacharyya and Kshirsagar lower bounds are compared with bootstrap methods for approximating the variance of the unbiased estimator of the reliability function in Burr XII.

Although, bootstrap is a simple and common method for approximating a statistics, but here the bootstrap approximations are below the lower bounds and this shows that the lower bounds are much more near to the exact value of the variance with respect to bootstrap.

As it seen in the Table 1 and Figure 1, we can conclude that for the less values of θ and a, the Kshirsagar bounds are the appropriate ones to approximate the variance of any unbiased estimator of $g(\theta)$ and for the high values, the Bhattacharyya and Kshirsagar bounds are not significantly different, so the Bhattacharyya bounds are the best because of their simple calculations.

Table 2. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the reliability function in Burr III

θ	b	B_1	B_2	B_3	B_4	B_5	$K_1 \approx K_2 \approx \ldots \approx K_5$
1	1	0.120113	0.171397	0.189380	0.193563	0.193761	0.250000
1	2	0.134105	0.161345	0.162755	0.164088	0.169406	0.444444
2	5	0.009908	0.016120	0.018066	0.018229	0.019653	0.945216
3	1	0.067563	0.067670	0.076366	0.083803	0.085934	0.765625

In Table 2 for Burr III, the Kshirsagar bounds were equal up to 5 decimal digits and their differences with Bhattacharyya bounds are noticeable. So, in this case, the Kshirsagar

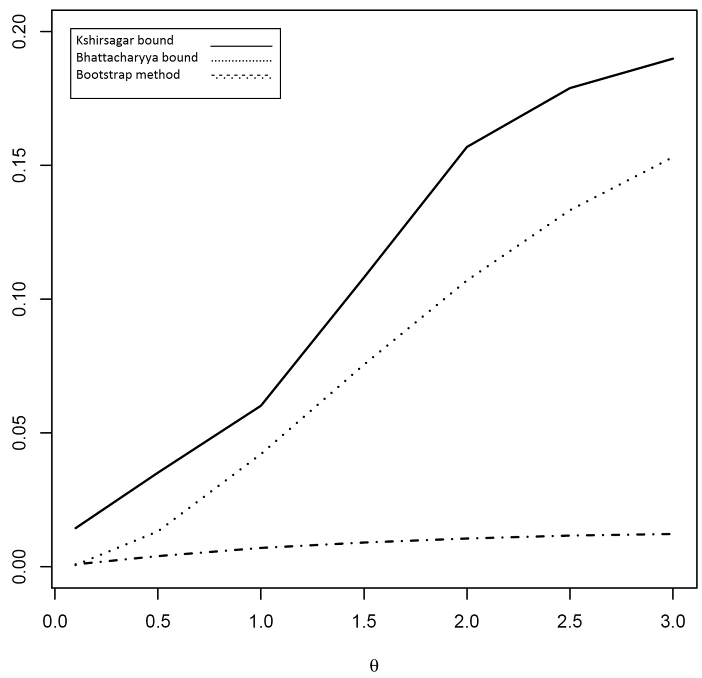

Figure 1. Comparing Bhattacharyya and Kshirsagar bounds and Bootstrap method (with 10000 replications) for the variance of any unbiased estimator of reliability function in Burr XII distribution with $a = 0.2$.

bound of order one is the best lower bound.

4.2 Lower bounds for the variance of any unbiased estimator of the hazard rate function in Burr XII and Burr III

As it is said before, the Burr XII distribution has a non-monotone hazard function and for different values of θ and α it can take many shapes of hazard function similar to other distribution. Thus, the use of this distribution as a failure model is more interesting. So, in reliability literature, estimating the hazard rate of Burr XII distribution has gathered the attention of many authors like, Evans and Ragab (1983), Basu and Ebrahimi (1991), Al-Hussaini and Jaheen (1992), Wingo (1993) and Soliman (2005).

The hazard rate of Burr XII distribution is,

$$h(x) = \frac{\alpha\theta x^{\alpha-1}}{1+x^{\alpha}}.$$

For $\alpha > 1$, the $h(x)$ has one critical point (single maximum) at $x = (\alpha-1)^{1/\alpha}$. It is clear that the height of $h(x)$ can be controlled by the parameter θ.

Since, only first derivation of $h(x)$ with respect to θ exists, we can calculate only the first Bhattacharyya bound which is equivalent to the Cramer-Rao bound. Also, in Kshirsagar bounds by supposing $\phi_i = \theta + i\delta$, we have for order k of Kshirsagar matrix,

$$Var_{\theta}(T(X)) \geq \sup_{\delta} \frac{kx^{2\alpha-2}\alpha^2\theta}{(1+x^{\alpha})^2}[(k+1)\delta+\theta],$$

where $\delta > -\frac{\theta}{k}$.

In Table 3, we present the Cramer-Rao bound and Kshirsagar bounds for the variance of any unbiased estimator of the hazard rate in Burr XII.

Table 3. First order Bhattacharyya bound (Cramer-Rao bound) and Kshirsagar bounds for the variance of any unbiased estimator of hazard rate in Burr XII

θ	x	α	B_1 = Cramer-Rao bound	K_1	K_2	K_3	K_4	K_5
0.1	0.2	0.3	0.00327	0.00327	0.00655	0.00982	0.01310	0.01638
0.5	2	1	0.02777	0.02777	0.05555	0.08333	0.11111	0.13888
3	2	4	31.88927	31.88927	63.77854	95.66782	127.55709	159.44636
4	6	2	1.68298	1.68298	3.36596	5.04894	6.73192	8.41490
6	10	5	8.99982	8.99982	17.99964	26.99946	35.99928	44.99910

The hazard rate function in Burr III distribution is as follow,

$$h(y) = \frac{\alpha\theta(y^{\alpha+1} + y)}{(1 + y^{-\alpha})^{\theta} - 1}.$$

In spite of Burr XII distribution, any order of Bhattacharyya bounds in Burr III can be evaluated. Table 4, shows the first five Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the hazard rate function in Burr III.

Table 4. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the hazard rate function in Burr III

θ	y	a	B_1	B_2	B_3	B_4	B_5
0.1	0.2	0.3	0.00201	0.00396	0.00584	0.00766	0.00942
0.5	2	1	0.00150	0.00291	0.00421	0.00541	0.00652
3	2	4	0.02748	0.05322	0.07729	0.09974	0.12063
θ	y	a	K_1	K_2	K_3	K_4	K_5
0.1	0.2	0.3	0.00212	0.00399	0.00605	0.00915	0.00998
0.5	2	1	0.00165	0.00312	0.00591	0.00646	0.00708
3	2	4	0.02758	0.06985	0.08050	0.13011	0.15151

As it is seen in Tables 3 and 4, the two bounds are significantly different in Burr XII, especially for some values of parameters , so it is recommended to use the Kshirsagar bounds of higher orders, which are more sharper. But in Burr III, the two bounds are not significantly different, so we use the first order Bhattacharyya bound, which has simpler evaluation.

4.3 Lower bounds for the variance of any unbiased estimator of the mode in Burr XII and Burr III

The density of Burr XII is unimodal at

$$Mode = \left(\frac{\alpha - 1}{\alpha\theta + 1}\right)^{\frac{1}{\alpha}}, \tag{8}$$

if $\alpha > 1$ and L-shaped if $\alpha \leq 1$ and the density of Burr III is also unimodal at

$$Mode = \left(\frac{\alpha + 1}{\alpha\theta - 1}\right)^{-\frac{1}{\alpha}}, \tag{9}$$

if $\alpha\theta > 1$ and L-shaped if $\alpha\theta \leq 1$.

The "average" of a sample of data or random variable can be quantified by the mean, median, or mode, with the mean used most often. Although these three measures of location coincide for symmetric distributions, they can differ markedly for observed data. The mode, however, is closer to the intuitive understanding of an "average" than are the mean and median since it is the value with the maximum probability.

Jones (1953) approximated the mode from weighted sample values. Bickel (2001, 2002) has obtained some estimation of mode.

$B_1, \ldots, B_5$ and $K_1, \ldots, K_5$ are the first five Bhattacharyya and Kshirsagar bounds for different values of α and θ in Burr XII and Burr III that are presented in Tables 5 and 6.

Table 5. Bhattacharyya and Kshirsagar bounds for variance of any unbiased estimator of the mode in Burr XII

θ	α	B_1	B_2	B_3	B_4	B_5
0.1	2	0.005781	0.010217	0.013625	0.016258	0.018303
1	2.5	0.041445	0.051807	0.055190	0.056589	0.057283
2	3	0.035412	0.041916	0.043923	0.044790	0.045248
4	6	0.014970	0.017869	0.018946	0.019485	0.019801
θ	α	K_1	K_2	K_3	K_4	K_5
0.1	2	0.026420	0.026509	0.026614	0.026619	0.026628
1	2.5	0.057731	0.057917	0.058574	0.058593	0.058668
2	3	0.045252	0.045407	0.046167	0.046186	0.046288
4	6	0.019868	0.019976	0.020605	0.020624	0.020737

Table 6. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the mode in Burr III

θ	α	B_1	B_2	B_3	B_4	B_5
1.5	1	0.56250	1.12500	1.68750	2.25000	2.81250
0.5	3	0.25000	0.25000	0.36111	0.38889	0.63889
3	2.5	0.34953	0.49896	0.59374	0.66166	0.71406
5	10	0.01403	0.01814	0.02015	0.02136	0.02217
θ	α	K_1	K_2	K_3	K_4	K_5
1.5	1	0.56275	1.15342	1.82300	2.42103	2.97650
0.5	3	0.26047	0.26471	0.37154	0.39910	0.68124
3	2.5	1.12136	1.14800	1.16320	1.17006	1.21556
5	10	0.02374	0.02401	0.02587	0.028440	0.03142

It is seen that the convergence of Kshirsagar bounds are the same as Bhattacharyya bounds.

4.4 Lower bounds for the variance of any unbiased estimator of the median in Burr XII and Burr III

Since the cdf of Burr XII and Burr III have closed forms, it is easy to see that their quantile x_q and y_q of order q are respectively as,

$$x_q = \left[(1-q)^{-\frac{1}{\theta}} - 1\right]^{\frac{1}{\alpha}},$$

$$y_q = \left[q^{-\frac{1}{\theta}} - 1\right]^{-\frac{1}{\alpha}}.$$

So, the median in Burr XII and Burr III distributions are obtained for $q = \frac{1}{2}$ as follow,

$$Median = \left[2^{\frac{1}{\theta}} - 1\right]^{\frac{1}{\alpha}},$$

$$Median = \left[2^{\frac{1}{\theta}} - 1\right]^{-\frac{1}{\alpha}}.$$

Since the mean is very sensitive to outliers and to long tails in the distribution, in many situations, statisticians use the median instead, which is generally much safer (Hampel et al., 2005). Ashour and El-Wakeel (1994) discussed Bayesian prediction of the median of the Burr distribution.

In Tables 7 and 8, we evaluate the first five Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the median in Burr XII and Burr III distributions for some values of θ and α.

Table 7. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the median in Burr XII

θ	α	B_1	B_2	B_3	B_4	B_5
0.7	0.2	11919.92	170386.1	981303.5	3035420	6024238
0.5	1	30.74899	45.52244	48.67708	49.05600	49.08512
2	0.5	0.164865	0.261329	0.269681	0.269988	0.269995
θ	α	K_1	K_2	K_3	K_4	K_5
0.7	0.2	6619864.3	9130395.3	10992791	11601268.1	12158103
0.5	1	48.24566	49.068000	49.086412	49.11458	49.11589
2	0.5	0.258225	0.269968	0.2699949	0.2711145	0.2711245

Furthermore, in Figure 2 we compare the first order Bhattacharyya and first order Kshirsagar lower bounds with the bootstrap approximation of the variance of the unbiased estimator of the median in Burr XII, which indicates that, with respect to the bootstrap approximation, the Bhattacharyya and Kshirsagar lower bounds are much more nearer to the exact value of the variance. This comparison shows that the two lower bounds are good approximations for the variance of the unbiased estimators.

Remark 4.1 It should be noted that when $\theta \to 0$, the median converge to the infinity and may be because of this, all our computation results for Bhattahcrayya bound, Kshirsagar bound and bootstrapping are tends to infinity when $\theta \to 0$. For example for $\theta = 0.1$ and $\alpha = 1$, we obtained $B_1 = 5.0381 \times 10^7$, $K_1 = 6.5613 \times 10^7$ and bootstrap with 100000 replications 3.901591×10^{14}.

Table 8. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the median in Burr III

θ	α	B_1	B_2	B_3	B_4	B_5
0.7	0.2	0.323065	6.10754	59.5259	382.2414	1832.327
0.5	1	0.379617	0.886251	1.409861	1.931078	2.451362
2	2	0.845060	1.276075	1.567792	1.788939	1.967084
θ	α	K_1	K_2	K_3	K_4	K_5
0.7	0.2	2.659874	1021.1254	2548.365	2987.6501	3124.254
0.5	1	1.025489	1.987485	2.15480	2.88875	2.999865
2	2	1.002547	1.8574210	2.369850	2.658741	2.659995

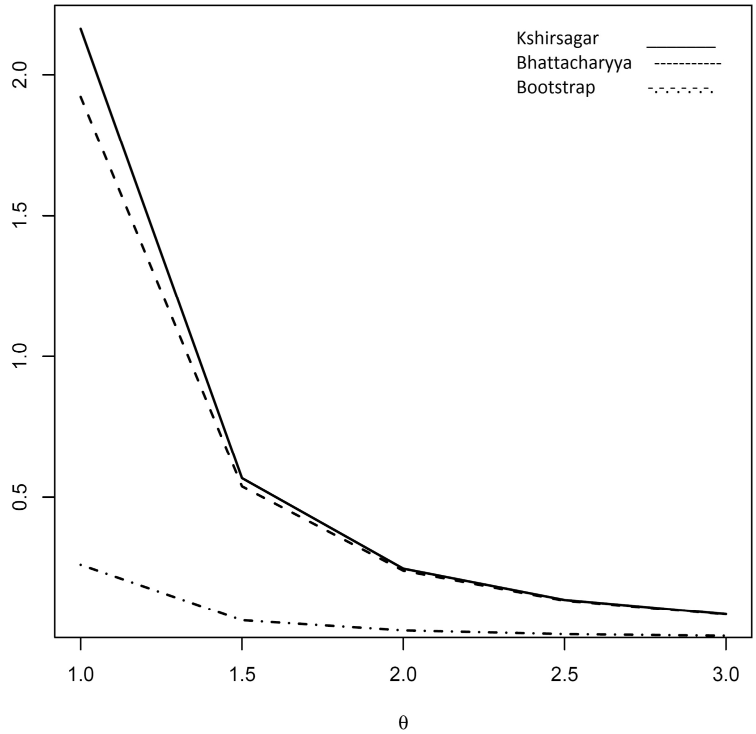

Figure 2. Comparing Bhattacharyya and Kshirsagar bounds of orders 1 and Bootstrap method (with 10000 replications) for the variance of any unbiased estimator of median in Burr XII distribution with $\alpha = 1$.

According to Tables 7 and 8, when the differences between Bhattacharyya and Kshirsagar bounds are not significant, we suggest using Bhattacharyya bounds because of their simple evaluations, otherwise, when the differences are significant, the Kshirsagar bounds are suggested to be used for their sharpness than Bhattacharyya bounds.

4.5 Lower bounds for the variance of any unbiased estimator of the mean function in Burr XII and Burr III

In this section, in addition to evaluating the Bhattacharyya and Kshirsagar bounds for different values of parameters in Burr XII and Burr III distributions, we evaluate the exact value of the variance of $T(X) = X$, which is the unbiased estimator of the mean. The results which are presented in Tables 9 and 10, show that the bounds are very close to the exact value of the variance of the estimator.

Table 9. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the mean function in Burr XII

θ	α	B_1	B_2	B_3	B_4	B_5	K_1	K_2	K_3	$Var(T(X))$
4	6	0.0213	0.0236	0.0246	0.0251	0.0254	0.0250	0.0251	0.0255	0.0263
1	3	0.8028	0.8893	0.9451	0.9484	0.9545	0.9398	0.9448	0.9550	0.9563
2	5	0.0555	0.0576	0.0592	0.0598	0.0602	0.0586	0.0587	0.0610	0.0615
4	1	0.1975	0.2193	0.2219	0.2221	0.2222	0.2222	0.2222	0.2222	0.2223
7	2	0.0392	0.0402	0.0407	0.0408	0.0409	0.0405	0.0405	0.0410	0.0411

Also, in this case, we compare the bounds with bootstrap method in Figure 3, which shows that the Bhattacharyya and Kshirsagar bounds can be used for the approximation of the variance as well as bootstrap.

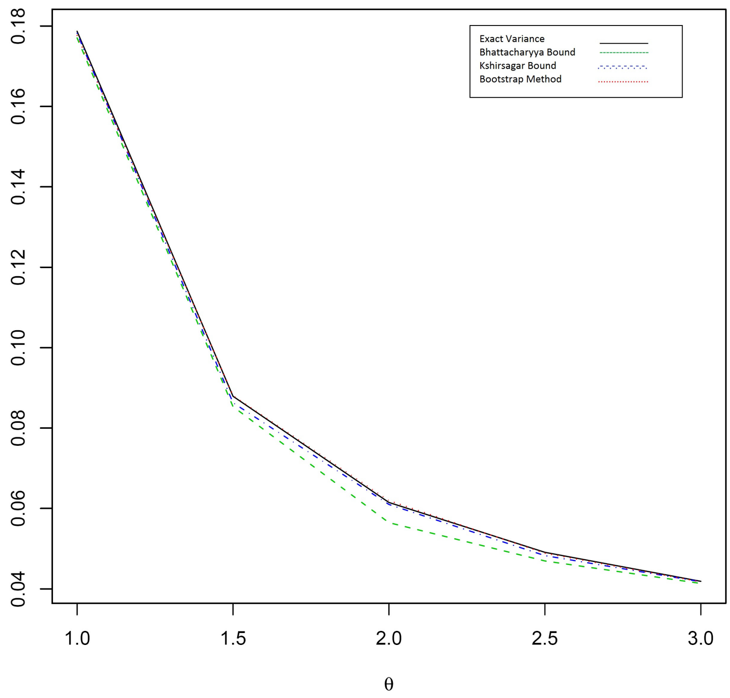

Figure 3. Comparing Bhattacharyya and Kshirsagar bounds of orders 3 and Bootstrap method (with 10000 replications) for the variance of any unbiased estimator of mean in Burr XII distribution with $\alpha = 5$.

Table 10. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the mean function in Burr III

θ	α	B_1	B_2	B_3	B_4	B_5	K_1	K_2	K_3	$Var(T(X))$
4	6	0.0667	0.0855	0.0959	0.1080	0.1157	0.1170	0.1186	0.1297	0.1383
1	5	0.0949	0.1176	0.1286	0.1354	0.14.88	0.1479	0.1499	0.1635	0.1787
2	3	0.3958	0.5486	0.6379	0.7002	0.8645	0.9853	1.0053	1.0655	1.4314
6	4	0.2521	0.3436	0.3946	0.4282	0.5031	0.5426	0.55179	0.5779	0.6885
7	3	0.7934	1.1301	1.3324	1.4725	2.1476	2.1732	2.2178	2.4526	3.1513

We can easily see that the Kshirsagar bounds, especially in Burr III, are sharper than the Bhattacharyya bounds and are very close to the exact values of variance of the unbiased estimator.

The evaluation of the Kshirsagar bounds of order greater than 3, are very difficult and time- consuming because of taking supremums, so we stopped at order 3.

5. Conclusion

In this paper, via comparing with the bootstrap method, we showed that the Bhattacharyya and Kshirsagar bounds are good approximations for the variance of any unbiased estimator of the parameter function $g(\theta)$ in Burr XII and Burr III distributions. We saw that, in estimating the hazard rate function in Burr III and mode function in both distributions, the convergence of the two bounds are approximately equal, so we used Bhattacharyya bounds, which have simpler evaluations. Furthermore, in both Burr XII and Burr III, the Kshirsagar bounds in estimating the mean, median, reliability and hazard rate (only Bur XII) functions (for some values of parameters), are more better than their corresponding Bhattacharyya bounds and the important problem is their evaluations, which nowadays can be easily done by computer and related softwares.

Also in the last section, for the mean function, we showed that how much the lower bounds may be converge and become close to the exact value of the variance.

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