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Estimation of the ratio, product and mean using multi auxiliary variables in the presence of non-response

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Abstract

This paper addresses the problem of estimating the population ratio, product and mean using multi auxiliary information in presence of non-response. Some classes of estimators have been proposed with their properties. Asymptotic optimum estimator(s) in the class(s) have been investigated along with their mean squared error formulae. Further the optimum value (depending upon population parameters) when replaced from sample values gives the estimators having the mean squared errors of the asymptotic optimum estimators. An empirical study is carried out in the support of the present study. Both theoretical and empirical findings are encouraging and in favour of the present study.

Keywords: Study variate · Auxiliary variate · Bias · Mean squared error · Non-response

Mathematics Subject Classification: Primary 62D05

1. Introduction

In survey sampling, it is well recognized that the use of auxiliary information results in substantial gain in efficiency over conventional estimators, which do not utilize such information. The problem of estimation of ratio, product and mean using single auxiliary character has been dealt to great extent by several authors including Singh (1965), Rao (1987), Bisht and Sisodia (1990), Naik and Gupta (1991), Upadhyaya and Singh (1999), Singh and Tailor (2005 a,b) and Singh et al. (2007). Further the problem has been extended by using supplementary information on additional auxiliary character by various authors such as Chand (1975), Sahoo and Sahoo (1993), Sahoo et al. (1993), Sahoo and Sahoo (1999) and Singh and Ruiz Espejo (2000).

Quite often information on many supplementary variables are available in the survey, which can be utilized to increase the precision of the estimate. Olkin (1958) has considered the use of multi auxiliary variables, positively correlated with the study variable to build up a multi variate ratio estimator of the population mean. Singh (1967) extended Olkin's estimator to the case where auxiliary variables are negatively correlated with variate under study. Later various authors including Shukla (1965, 1966), Mohanty (1967), Tujeta and Bahl (1991) and Agrawal and Panda (1993, 1994) have used the information on several auxiliary variables in building up estimators for population mean. Khare (1991) has suggested a generalized class of estimators for estimating the ratio of two means using multi

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auxiliary characters with known population means.

It is well known especially in human surveys that information is generally not obtained from all the sample units even after callbacks. The problem of estimating the parameters such as ratio of two means, population mean and variance when some observations are missing due to random non response has been discussed by Toutenberg and Srivastava (1998), Singh and Joarder (1998), Singh S. et al. (2000), Singh and Tracy (2001) and Singh H. P. et al. (2003). In case of non-random non-response, the problem of estimation of population mean using information on single auxiliary character has been considered by different authors such as El Badry (1956), Srinath (1971), Cochran (1977), Rao (1986, 1987), Khare and Srivastava (1993, 1995, 1997), Tabasum and Khan (2004); Tabasum and Khan (2006), Khare and Sinha (2004, 2007), Singh and Kumar (2008 a,b, 2009 a,b, 2010, 2011), Kumar et al. (2011) and Gamrot (2011) have discussed the problem of estimating the ratio of two means using multi auxiliary characters in the presence of non-response.

In this paper I have suggested some classes of estimators for ratio, product and mean using multi auxiliary in different situations and their properties have been studied. Conditions for attaining minimum mean squared error of the proposed classes of estimators have also been obtained. Estimators based on estimated optimum values have been obtained with their approximate mean squared error. An empirical study has been carried out in support of the present study.

2. Notations and sampling procedure

Let y_{il} $(i = 0, 1)$ and x_{jl} $(j = 1, 2, ..., p)$ be the non-negative values of l^{th} unit of the study variate y_i $(i = 0, 1)$ and the auxiliary variates x_j $(j = 1, 2, ..., p)$ for a population of size N with population means $\overline{Y}_i$ $(i = 0, 1)$ and $\overline{X}_j$ $(j = 1, 2, ..., p)$. When non-response occurs, the subsampling procedure of Hansen and Hurwitz (1946) is an alternative to call backs and similar procedures. In this approach, the population of size N is assumed to be composed of two strata of size N_1 and $N_2 = N - N_1$, of "respondents" and "non-respondents". The initial simple random sample of size n is drawn without replacement results in n_1 respondents and n_2 non-respondents. A sub sample of size $m = n_2/k$, where $(k > 1)$ is predetermined, is drawn from the n_2 non-respondents and through intensive efforts information on the study variates y_i $(i = 0, 1)$ are assumed to be obtained from all of the m units (see, Rao (1983)). Thus the estimator for the population mean Y_i $(i = 0, 1)$ of the finite population is

$$\overline{y}_i^* = (n_1/n)\overline{y}_{i(1)} + (n_2/n)\overline{y}_{i(2)} \ , i = 0, 1 \tag{2.1}$$

where $\overline{y}_{i(1)}$ and $\overline{y}_{i(2)}$;$i = 0, 1$ are the sample means of the characters $\overline{y}_i$ $(i = 0, 1)$ based on n_1 and m units respectively. The estimator $\overline{y}_i^*$ is unbiased and has variance

$$Var\left(\overline{y}_i^*\right) = \left(\frac{1-f}{n}\right) S_{y_i}^2 + \frac{W_2(k-1)}{n} S_{y_i(2)}^2 \tag{2.2}$$

where $f = n/N$, $W_2 = N_2/N$, $S_{y_i}^2$ and $S_{y_i(2)}^2$ are the population mean square of the variates y_i $(i = 0, 1)$ for the entire population and for non-responding group of the population.

Similarly the estimator $\overline{x}_j^*$ $(j = 1, 2, ...p)$ for the population mean $\overline{X}_j$ is given by

$$\overline{x}_j^* = (n_1/n)\overline{x}_{j(1)} + (n_2/n)\overline{x}_{j(2)} \tag{2.3}$$

The estimator $\overline{x}_j^*$ $(j = 1, 2, ..., p)$ is unbiased an has te variance

$$Var\left(\overline{x}_j^*\right) = \left(\frac{1-f}{n}\right) S_{x_j}^2 + \frac{W_2(k-1)}{n} S_{x_j(2)}^2 \tag{2.4}$$

where $S_{x_j}^2$ and $S_{x_j(2)}^2$ $(j = 1, 2, ..., p)$ are the population mean square of x_j for the entire population and non responding group of the population.

Let $\hat{R}_{(\alpha)}^* = (\overline{y}_0^*/\overline{y}_1^{*\alpha})$, $(\overline{y}_1^* \neq 0)$ denote the conventional estimator of the population parameter $R_{(\alpha)} = \left(\overline{Y}_0/\overline{Y}_1^{\alpha}\right)$, $\overline{Y}_1 \neq 0$, α being a constant takes values (1,-1,0). For different values of α, the following holds

(i) for $\alpha = 1$, $\hat{R}_{(\alpha)}^* \longrightarrow \hat{R}_{(1)}^* = \frac{\overline{y}_0^*}{\overline{y}_1^*} = \hat{R}^*$ (say) is the conventional estimator of the ratio $R_{(\alpha)} \longrightarrow R_{(1)} = \left(\overline{Y}_0/\overline{Y}_1\right) = R$ (say).

(ii) for $\alpha = -1$, $\hat{R}_{(\alpha)}^* \longrightarrow \hat{R}_{(-1)}^* = \overline{y}_0^* \overline{y}_1^* = \hat{P}^*$ (say) is the conventional estimator of the ratio $R_{(\alpha)} \longrightarrow R_{(-1)} = \overline{Y}_0 \overline{Y}_1 = P$ (say).

(iii) for $\alpha = 0$, $\hat{R}_{(\alpha)}^* \longrightarrow \hat{R}_{(0)}^* = \overline{y}_0^*$, is the conventional estimator of the population mean $\overline{Y}_0$.

Let $u_j = \frac{\overline{x}_j^*}{\overline{X}_j}$, for $j = 1, 2, ..., p = \frac{\overline{x}_{j-p}^*}{\overline{X}_{j-p}}$, for $j = p+1, p+2, ..., 2p$ and u denotes the column vector of $2p$ elements $u_1, u_2, ..., u_{2p}$. Super fix T over a column vector denotes the corresponding row vector. Defining

$\overline{y}_0^* = \overline{Y}_0(1+\eta_0)$, $\overline{y}_1^* = \overline{Y}_1(1+\eta_1)$, $\varepsilon_0 = \left\{\left(\hat{R}_{\alpha}^*/R_{\alpha}^*\right) - 1\right\} \approx \left(\eta_0 - \alpha\eta_1 - \alpha^2\eta_1^2 - \alpha\eta_0\eta_1\right)$,
$\varepsilon_j = (u_j - 1)$, $j = 1, 2, ..., 2p$ and let $\varepsilon^T = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2p})$.

Then to the first degree of approximation, the following holds

$E(\varepsilon_0) = \alpha\left[\left(\frac{1-f}{n}\right) C_{y_1}\left(\alpha C_{y_1} - \rho_{y_0y_1} C_{y_0}\right) + \frac{W_2(k-1)}{n} C_{y_1(2)}\left(\alpha C_{y_1(2)} - \rho_{y_0y_1(2)} C_{y_0(2)}\right)\right]$,
$E(\varepsilon_j) = 0 \,\forall\, j = 1, 2, ..., 2p$,
$E(\varepsilon_0\varepsilon_j) = E\left[(\eta_0 - \alpha\eta_1)\varepsilon_j\right] = \left[\left(\frac{1-f}{n}\right) q_{(\alpha)j} + \frac{W_2(k-1)}{n} q_{(\alpha)j}^{(2)}\right]$, $j = 1, 2, ..., p$,
$E(\varepsilon_0\varepsilon_j) = \left(\frac{1-f}{n}\right) q_{(\alpha)j}$, $j = p+1, p+2, ..., 2p$,
$E(\varepsilon_j\varepsilon_l) = \left[\left(\frac{1-f}{n}\right) a_{jl} + \frac{W_2(k-1)}{n} a_{jl}^{(2)}\right] = e_{jl}$(say), $(j, l) = 1, 2, ..., p$,
$\quad = \left(\frac{1-f}{n}\right) a_{jl} = f_{jl}$ (say), $(j, l) = 1, 2, ..., p, p+1, ..., 2p$

where
$a_{jl} = \rho_{x_jx_l} C_{x_j} C_{x_l}$, $a_{jl}^{(2)} = \rho_{x_jx_l(2)} C_{x_j(2)} C_{x_l(2)}$, $C_{y_i}^2 = S_{y_i}^2/\overline{Y}_i^2$, $C_{y_i(2)}^2 = S_{y_i(2)}^2/\overline{Y}_i^2$, $i = 0, 1$,
$C_{x_j}^2 = S_{x_j}^2/\overline{X}_j^2$, $C_{x_j(2)}^2 = S_{x_j(2)}^2/\overline{X}_j^2$, $j = 1, 2, ..., p$,
$q_{(\alpha)j} = C_{x_j}\left(\rho_{y_0x_j} C_{y_0} - \alpha\rho_{y_1x_j} C_{y_1}\right)$, $q_{(\alpha)j}^{(2)} = C_{x_j(2)}\left(\rho_{y_0x_j(2)} C_{y_0(2)} - \alpha\rho_{y_1x_j(2)} C_{y_1(2)}\right)$, $j = 1, 2, ..., p$,
$\left(\rho_{y_0y_1}, \rho_{y_ix_j}, \rho_{x_jx_l}, \ i = 0, 1; (j,\ l) = 1, 2, ..., p\right)$ and $\left(\rho_{y_0y_1(2)}, \rho_{y_ix_j(2)}, \rho_{x_jx_l(2)}, \ i = 0, 1; (j,\ l) = 1, 2, ..., p\right)$ are the correlation coefficients between $(y_0\ ,\ y_1)$, $(y_i\ ,\ x_j)$ and $(x_j\ ,\ x_l)$ respectively for the entire population and for the non-responding group of the population.

$$
\begin{aligned}
b_{(\alpha)}^{T} &= \left(a_{(\alpha)}^{T}, C_{(\alpha)}^{T}\right) = \left(a_{(\alpha)1}, a_{(\alpha)2}, ..., a_{(\alpha)p}, C_{(\alpha)1}, C_{(\alpha)2}, ..., C_{(\alpha)p}\right) \\
a_{(\alpha)}^{T} &= \left[\left(\frac{1-f}{n}\right) q_{(\alpha)}^{T} + \frac{W_2\,(k-1)}{n}\left(q_{(\alpha)}^{(2)}\right)^{T}\right],\ q_{(\alpha)}^{T} = \left(q_{(\alpha)1}, q_{(\alpha)2}, ..., q_{(\alpha)p}\right), \\
\left(q_{(\alpha)}^{(2)}\right)^{T} &= \left(q_{(\alpha)1}^{(2)},\ q_{(\alpha)2}^{(2)}, ..., q_{(\alpha)p}^{(2)}\right),\ a_{(\alpha)j} = \left[\left(\frac{1-f}{n}\right) q_{(\alpha)j} + \frac{W_2\,(k-1)}{n} q_{(\alpha)j}^{(2)}\right],\ j = 1, 2, ..., p\ , \\
C_{(\alpha)}^{T} &= \left(\frac{1-f}{n}\right) q_{(\alpha)}^{T}, C_{(\alpha)j} = \left(\frac{1-f}{n}\right) q_{(\alpha)j},\ j = 1, 2, ..., p,\ D = \begin{bmatrix} E & F \\ F^{T} & F \end{bmatrix}
\end{aligned}
$$

which is assumed to be positive definite. The matrices $E = (e_{jl})_{p\times p}$ and $F = (f_{jl})_{p\times p}$ are $p \times p$ matrices. Now utilizing the multi auxiliary characters with known population means, I have suggested a class of estimators for the parameter $R_{(\alpha)}$ in section 3.

3. The class of estimators

Suppose non-response occurs on the study variables $(y_0,\ y_1)$, information on the p-auxiliary variables $\overline{x}_j,\ j = 1,\ 2,\ ...,\ p$ are obtained from all sample units (i.e. the initial sample units), and the population means $\overline{X}_j,\ j = 1,\ 2,\ ...,\ p$ of p-auxiliary variables are known. In this situation we note that when suggesting the estimator for the population parameter $R_{(\alpha)}$, Khare and Sinha (2007) used only the information on the sample means $\overline{x}_j,\ j = 1,\ 2,\ ...,\ p$ and on the population means $\overline{X}_j,\ j = 1,\ 2,\ ...,\ p$ of the p-auxiliary variables $\overline{x}_j,\ j = 1,\ 2,\ ...,\ p$. However one can also obtain the unbiased estimators $\overline{x}_j^{*} = (n_1/n)\,\overline{x}_{j(1)} + (n_2/n)\,\overline{x}_{j(2)}$ of the population mean $\overline{X}_j,\ j = 1,\ 2,\ ...,\ p$ (without any extra effort) while in the process of obtaining $\overline{y}_i^{*} = (n_1/n)\,\overline{y}_{i(1)} + (n_2/n)\,\overline{y}_{i(2)},\ i = 0,\ 1$ the unbiased estimators of the population means $\overline{Y}_i\ ,\ (i = 0,\ 1)$. Thus, in the situation stated above we have two unbiased estimators $\overline{x}_j^{*}$ and $\overline{x}_j$ of the population mean $\overline{X}_j,\ j = 1,\ 2,\ ...,\ p$ of the auxiliary variate $x_j,\ j = 1,\ 2,\ ...,\ p$. With this background author convinced to suggest the class of estimators, $G_{(\alpha)} = G\left(\hat{R}_{(\alpha)}^{*}, u_1, u_2, ..., u_{2p}\right) = G\left(\hat{R}_{(\alpha)}^{*}, u^{T}\right)$ of the population parameter $R_{(\alpha)}$.

Let e^{T} denote the row vector of $2p$ unit elements. Whatever be the sample chosen, let $\left(\hat{R}_{(\alpha)}^{*}, u^{T}\right)$ assume values in a closed convex subset, S of the $(2p+1)$ dimensional real space containing the point $\left(R_{(\alpha)}, e^{T}\right)$. Let $G\left(\hat{R}_{(\alpha)}^{*}, u^{T}\right)$ be a function of $\left(\hat{R}_{(\alpha)}^{*}, u_1, u_2, ..., u_{2p}\right)$ such that

$$
G\left(R_{(\alpha)}, e^{T}\right) = R_{(\alpha)} \text{ for all } R_{(\alpha)} \tag{3.1}
$$

and which is continuous and bounded with continuous and bounded first and second order partial derivatives in S.
Define a class of estimators of the parameter $R_{(\alpha)}$ as

$$
G_{(\alpha)} = G\left(\hat{R}_{(\alpha)}^{*}, u_1, u_2, ..., u_{2p}\right) = G\left(\hat{R}_{(\alpha)}^{*}, u^{T}\right) \tag{3.2}
$$

Since there are only a finite number of samples, the expectations and mean squared error of the estimator $G_{(\alpha)}$ exist under the above conditions.

To obtain the mean squared error of $G_{(\alpha)}$, expand the function $G\left(\hat{R}^*_{(\alpha)}, u^T\right)$ in a second order Taylor's series

$$\begin{aligned}G_{(\alpha)} &= G\left(R_{(\alpha)}, e^T\right) + \left(\hat{R}^*_{(\alpha)} - R_{(\alpha)}\right) \left.\frac{\partial G(\,.\,)}{\partial \hat{R}^*_{(\alpha)}}\right|_{\left(R_{(\alpha)}, e^T\right)} + (u-e)^T G^{(1)}\left(R_{(\alpha)}, e^T\right) \\ &+ \frac{1}{2}\left\{\left(\hat{R}^*_{(\alpha)} - R_{(\alpha)}\right)^2 \left.\frac{\partial^2 G(\,.\,)}{\partial \hat{R}^{*2}_{(\alpha)}}\right|_{\left(\hat{R}^{**}_{(\alpha)}, u^{*T}\right)} + 2\left(\hat{R}^*_{(\alpha)} - R_{(\alpha)}\right)(u-e)^T \left.\frac{\partial G^{(1)}(\,.\,)}{\partial \hat{R}^*_{(\alpha)}}\right|_{\left(\hat{R}^{**}_{(\alpha)}, u^{*T}\right)} \right. \\ &\left. + (u-e)^T G^{(2)}\left(\hat{R}^{**}_{(\alpha)}, u^{*T}\right)(u-e)\right\},\end{aligned}$$

where $\hat{R}^{**}_{(\alpha)} = R_{(\alpha)} + \eta\left(\hat{R}^*_{(\alpha)} - R_{(\alpha)}\right)$, $u^* = e + \eta(u-e)$, $0 < \eta < 1$; $G^{(1)}$ denotes the $2p$ elements column vector of first partial derivatives of $G(\,.\,)$ and $G^{(2)}$ denotes $2p \times 2p$ matrix of second partial derivatives of $G(\,.\,)$ with respect to u. Substituting for $\hat{R}^*_{(\alpha)}$ and u in terms of η_0, η_1, ε_0 and ε and using (3.1), one can get

$$\begin{aligned}G_{(\alpha)} =& R_{(\alpha)} + R_{(\alpha)}\left\{(1+\eta_0)(1+\eta_1)^{-\alpha} - 1\right\} \left.\frac{\partial G(\,.\,)}{\partial \hat{R}^*_{(\alpha)}}\right|_{\left(R_{(\alpha)}, e^T\right)} \\ &+ \varepsilon^T G^{(1)}\left(R_{(\alpha)}, e^T\right) \\ &+ \frac{1}{2}\left[\left\{(1+\eta_0)(1+\eta_1)^{-\alpha} - 1\right\}^2 \left.\frac{\partial^2 G(\,.\,)}{\partial \hat{R}^{*2}_{(\alpha)}}\right|_{\left(\hat{R}^{**}_{(\alpha)}, u^{*T}\right)}\right. \\ &+ 2R_{(\alpha)}\left\{(1+\eta_0)(1+\eta_1)^{-\alpha} - 1\right\}\varepsilon^T \left.\frac{\partial G^{(1)}(\,.\,)}{\partial \hat{R}^*_{(\alpha)}}\right|_{\left(\hat{R}^{**}_{(\alpha)}, u^{*T}\right)} \\ &\left. + \varepsilon^T G^{(2)}\left(\hat{R}^{**}_{(\alpha)}, u^{*T}\right)\varepsilon\right] . \end{aligned} \tag{3.3}$$

Taking expectation in (3.3) and noting that the second partial derivatives are bounded, the following theorem holds.

Theorem 3.1

$$E\left(G_{(\alpha)}\right) = R_{(\alpha)} + o\left(n^{-1}\right)$$

From theorem 3.1, it follows that the bias of the estimator $G_{(\alpha)}$ is of the order n^{-1}, and hence its contribution to the mean squared error of $G_{(\alpha)}$ will be of the order n^{-2}.

Now prove the following result

Theorem 3.2 To the first degree of approximation, the mean squared error of

$$G^{(1)}\left(R_{(\alpha)}, e^T\right) = -R_{(\alpha)} D^{-1} b_{(\alpha)} \tag{3.4}$$

and the minimum mean squared error is given by

$$\min. MSE\left(G_{(\alpha)}\right) = MSE\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)} b^T_{(\alpha)} D^{-1} b_{(\alpha)} \tag{3.5}$$

where

$$\begin{aligned} MSE\left(\hat{R}^*_{(\alpha)}\right) = \hat{R}^2_{(\alpha)} &\left[\left(\tfrac{1-f}{n}\right)\left(C^2_{y_0} + \alpha^2 C^2_{y_1} - 2\alpha\rho C_{y_0} C_{y_1}\right)\right. \\ &\left.+ \tfrac{W_2(k-1)}{n}\left(C^2_{y_0(2)} + \alpha^2 C^2_{y_1(2)} - 2\alpha\rho C_{y_0(2)} C_{y_1(2)}\right)\right] \end{aligned} \tag{3.6}$$

is the mean squared error of $\hat{R}^*_{(\alpha)}$ to the first degree of approximation.

PROOF From (3.3), the $MSE\left(G_{(\alpha)}\right)$ to the first degree of approximation is given by

$$\begin{aligned} MSE\left(G_{(\alpha)}\right) &= E\left(G_{(\alpha)} - R_{(\alpha)}\right)^2 \\ &= E\left[R_{(\alpha)}\left(\eta_0 - \alpha\eta_1\right) \left.\frac{\partial G(\ .\ )}{\partial \hat{R}^*_{(\alpha)}}\right|_{\left(R_{(\alpha)}, e^T\right)} + \varepsilon^T G^{(1)}\left(R_{(\alpha)}, e^T\right)\right]^2 \end{aligned} \tag{3.7}$$

From (3.1) which implies that $\left.\frac{\partial G(\ .\ )}{\partial \hat{R}^?_{(\alpha)}}\right|_{\left(R_{(\alpha)}, e^T\right)} = 1.$

Thus the expression (3.7) reduces to

$$\begin{aligned} \text{MSE}\left(G_{(\alpha)}\right) &= E\left[R_{(\alpha)}\left(\eta_0 - \alpha\eta_1\right) + \varepsilon^T G^{(1)}\left(R_{(\alpha)}, e^T\right)\right]^2, \\ &= E\left[R^2_{(\alpha)}\left(\eta_0 - \alpha\eta_1\right)^2 + 2R_{(\alpha)}\left(\eta_0 - \alpha\eta_1\right)\varepsilon^T G^{(1)}\left(R_{(\alpha)}, e^T\right)\right. \\ &\quad \left. + \left(G^{(1)}\left(R_{(\alpha)}, e^T\right)\right)^T \varepsilon\varepsilon^T G^{(1)}\left(R_{(\alpha)}, e^T\right)\right], \\ &= \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) + 2R_{(\alpha)} b^T_{(\alpha)} G^{(1)}\left(R_{(\alpha)}, e^T\right) + \left(G^{(1)}\left(R_{(\alpha)}, e^T\right)\right)^T D\left(G^{(1)}\left(R_{(\alpha)}, e^T\right)\right) \end{aligned} \tag{3.8}$$

which is minimized for

$$G^{(1)}\left(R_{(\alpha)}, e^T\right) = -R_{(\alpha)} D^{-1} b_{(\alpha)} \tag{3.9}$$

Substituting (3.9) in (3.8), the resulting (minimum) mean squared error of $G_{(\alpha)}$

$$\min.\text{MSE}\left(G_{(\alpha)}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)} b^T_{(\alpha)} D^{-1} b_{(\alpha)} \tag{3.10}$$

Thus the theorem is proved. ■

REMARK 3.1 The class of estimators $G_{(\alpha)}$ at (3.2) is very large. If the parameters in the function $G\left(\hat{R}_{(\alpha)}, u^T\right)$ are so chosen that they satisfy (3.4), the resulting estimator will have MSE given by (3.5). A few examples are:

(i) $G_{(1)} = \hat{R}^*_{(\alpha)} \exp\left\{\alpha^T \log u\right\}$, (ii) $G_{(2)} = \hat{R}^*_{(\alpha)}\left[1 + \varphi^T(u - e)\right]$,
(iii) $G_{(3)} = \hat{R}^*_{(\alpha)} \exp\left\{\varphi^T(u - e)\right\}$ (iv) $G_{(4)} = \hat{R}^*_{(\alpha)} + \varphi^T(u - e)$
(v) $G_{(5)} = \hat{R}^*_{(\alpha)} / \left\{\hat{R}^*_{(\alpha)} - \varphi^T(u - e)\right\}$

where $\varphi^T = (\varphi_1, \varphi_2, ..., \varphi_{2p})$ is a vector of $2p$ constants. The optimum values of these constants are obtained from (3.4). Since (3.4) involves $2p$ equations, taken exactly $2p$ unknown constants in defining above estimators of the class.

4. Estimator based on estimated optimum value

To obtain the estimator based on estimated optimum, adopt the same procedure as discussed in Singh (1982) and Srivastava and Jhajj (1983).

It is to be mentioned that the proposed class of estimator $G_{(\alpha)}$ at (3.2) will attained minimum MSE given by (3.5) (or (3.10)) only when the optimum value of the derivatives (or constants involved in the estimators) given by (3.4), which are functions of the unknown population parameters are used. To use such estimators in practice, one has to use some guessed values of the parameters in (3.4), either through past experience or through a pilot sample survey. It may be noted that even if the values of the constants used in the estimator are not exactly equal to their optimum values as given by (3.4) (or (3.9)) but are close enough, the resulting estimator will be better than usual estimator $\hat{R}^*_{(\alpha)}$ as has been demonstrated by Das and Tripathi (1978). For more discussion on this point in connection with the estimation of population mean the reader is referred to Srivastava (1966), Murthy (1967, p.325), Reddy (1973, 1974) and Srivenkataramana and Tracy (1980).

However there are situations where the exact optimum values of the derivative given by (3.4) or its guessed value may be rarely known in practice, hence it is advisable to replace it by its estimate from sample values. We suppose that the equation (3.4) can be solved uniquely for the $2p$ unknown constants in the estimator (3.2). The optimum values of these constants will involve $D^{-1}b_{(\alpha)}$ or may be both $D^{-1}b_{(\alpha)}$ and $R_{(\alpha)}$, which are unknown. When these optimum values are inserted in (3.2), it no longer remains an estimator since it involves unknown $\psi = D^{-1}b_{(\alpha)}$, and may be also $R_{(\alpha)}$. Let $\hat{\psi}$ be a consistent estimator of ψ computed from the sample data at hand. Then replace ψ by $\hat{\psi}$ and also $R_{(\alpha)}$ by $\hat{R}^*_{(\alpha)}$ if necessary, in the optimum $G_{(\alpha)}$ resulting in the estimator $G^*_{(\alpha)}$ say, which will now be a function of $\hat{R}^*_{(\alpha)}$, u and $\hat{\psi}$. Define

$$G^*_{(\alpha)} = G^*\left(\hat{R}^*_{(\alpha)},\ u^T,\ \hat{\psi}^T\right) \tag{4.1}$$

where the function $G^*\left(\hat{R}^*_{(\alpha)},\ u^T,\ \hat{\psi}^T\right)$ is derived from the function $G\left(\hat{R}_{(\alpha)},\ u^T\right)$ cited at (3.2) by replacing the unknown constants in it by the consistent estimates. The condition (3.1) will then imply that

$$G^*\left(\hat{R}^*_{(\alpha)},\ e^T,\ b^T\right) = R_{(\alpha)} \text{ for all } R_{(\alpha)}, \tag{4.2}$$

which in turns implies

$$\left.\frac{\partial G^*(\ .\ )}{\partial \hat{R}^*_{(\alpha)}}\right|_{(R_{(\alpha)},e^T,\ \psi^T)} = 1 \tag{4.3}$$

We further assume that

$$\frac{\partial G^*(\ .\ )}{\partial u}|_{(R_{(\alpha)},e^T,\ \psi^T)} = \frac{\partial G(\ .\ )}{\partial u}|_{(R_{(\alpha)},e^T,\ \psi^T)} = -R_{(\alpha)}\psi \tag{4.4}$$

and

$$\left.\frac{\partial G^*(\,.\,)}{\partial \hat{\psi}}\right|_{(R_{(\alpha)}, e^T,\, \psi^T)} = 0 \tag{4.5}$$

Expanding the function $G^*\left(R_{(\alpha)},\ u^T,\ \psi^T\right)$ about the point $\left(R_{(\alpha)},\ e^T,\ \psi^T\right)$ in a Taylor's series and using (4.1) to (4.5), one get

$$\begin{aligned}
G^*_{(\alpha)} &= G^*\left(R_{(\alpha)}, e^T, \psi^T\right) + \left(\hat{R}^*_{(\alpha)} - R_{(\alpha)}\right) \left.\frac{\partial G^*(\,.\,)}{\partial \hat{R}^*_{(\alpha)}}\right|_{(R_{(\alpha)}, e^T, \psi^T)} + (u-e)^T \left.\frac{\partial G^*(\,.\,)}{\partial u}\right|_{(R_{(\alpha)}, e^T, \psi^T)} \\
&\quad + \left(\hat{\psi} - \psi\right)^T \left.\frac{\partial G^*(\,.\,)}{\partial \hat{\psi}}\right|_{(R_{(\alpha)}, e^T, \psi^T)} + \text{ second order terms}, \\
&= R_{(\alpha)} + R_{(\alpha)}\varepsilon_0 + \varepsilon^T\left(-R_{(\alpha)}\psi\right) + \text{ second order terms}.
\end{aligned} \tag{4.6}$$

Since $\hat{\psi}$ is a consistent estimator of ψ, the expectation of the second order terms in (4.6) will be $o\left(n^{-1}\right)$ and hence

$$E\left(G^*_{(\alpha)}\right) = R_{(\alpha)} + o\left(n^{-1}\right)$$

From (4.6) one obtain

$$\left(G^*_{(\alpha)} - R_{(\alpha)}\right) = R_{(\alpha)}\left(\eta_0 - \alpha\eta_1\right) - R_{(\alpha)}\varepsilon^T\psi + \text{ second order terms}. \tag{4.7}$$

Squaring both sides of (4.7) and neglecting terms of $\varepsilon' s$ having power greater than two

$$\left(G^*_{(\alpha)} - R_{(\alpha)}\right)^2 = R^2_{(\alpha)}\left(\eta_0 - \alpha\eta_1\right)^2 + R^2_{(\alpha)}\psi^T\varepsilon\varepsilon^T\psi - 2R^2_{(\alpha)}\left(\eta_0 - \alpha\eta_1\right)\varepsilon^T\psi$$

or

$$\left(G^*_{(\alpha)} - R_{(\alpha)}\right)^2 = R^2_{(\alpha)}\left[\left(\eta_0 - \alpha\eta_1\right)^2 + \psi^T\varepsilon\varepsilon^T\psi - 2\left(\eta_0 - \alpha\eta_1\right)\varepsilon^T\psi\right] \tag{4.8}$$

Taking expectations of both sides in (4.8) one get the mean squared error of $G^*_{(\alpha)}$, to the first degree of approximation as

$$\text{MSE}\left(G^*_{(\alpha)}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}b^T_{(\alpha)}D^{-1}b_{(\alpha)}$$

which is same as given in (3.5) (or (3.10)) i.e.

$$\text{MSE}\left(G^*_{(\alpha)}\right) = \text{min.MSE}\left(G_{(\alpha)}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}b^T_{(\alpha)}D^{-1}b_{(\alpha)} \tag{4.9}$$

It may be noted that the following estimators:

(i) $d^*_{(1)} = \hat{R}^*_{(\alpha)}\exp\left\{\hat{\psi}^T \log u\right\}$ (ii) $d^*_{(2)} = \hat{R}^*_{(\alpha)}\left[1 - \hat{\psi}^T (u-e)\right]$,

(iii) $d^*_{(3)} = \hat{R}^*_{(\alpha)}\exp\left\{-\hat{\psi}^T (u-e)\right\}$, (iv) $d^*_{(4)} = \hat{R}^*_{(\alpha)} - \hat{\psi}^T (u-e)$,

(v) $d^*_{(5)} = \hat{R}^{*^2}_{(\alpha)}/\left\{\hat{R}^*_{(\alpha)} + \psi^T (u-e)\right\}$, etc.

are the members of the suggested class of estimators $G^*_{(\alpha)}$. It can be shown to the first degree of approximation that the mean squared errors of the estimators $G^*_{(j)}$, $j = 1$ to 5 are same and equals to the MSE $\left(G^*_{(\alpha)}\right) = \min.\text{MSE}\left(G_{(\alpha)}\right)$ given by (4.9).

For different values of α one can obtain a class of estimators for ratio, product and population mean for $G^*_{(\alpha)}$, $H_{(\alpha)}$, $F_{(\alpha)}$ and $J_{(\alpha)}$ respectively. The results are explained in the Appendix I, II, III and IV, respectively.

Remark 4.1 Population means $\overline{X}_1, \overline{X}_2, ..., \overline{X}_p$ are known, incomplete information on the study variates (y_0, y_1) and on the auxiliary variates $x_j\,(j = 1, 2, ..., p)$.
In this case we use information on $(n_1 + m)$ responding units on the study variates (y_0, y_1) and the auxiliary variate $x_j\,(j = 1, 2, ..., p)$ from the sample of size n along with known population means $\overline{X}_1, \overline{X}_2, ..., \overline{X}_p$. Thus propose a class of estimators for $R_{(\alpha)}$ as

$$H_{(\alpha)} = H\left(\hat{R}^*_{(\alpha)},\ \nu^T\right) \tag{4.10}$$

where ν denotes the column vector of p elements $\nu_1, \nu_2, ..., \nu_p$ with $\nu_j = \overline{x}^*_j/\overline{X}_j$, $j = 1, 2, ..., p$; $H\left(\hat{R}^*_{(\alpha)},\ \nu^T\right)$ is a function of $\left(\hat{R}^*_{(\alpha)},\ \nu^T\right)$ such that

$$H\left(R_{(\alpha)}, e^T\right) = R_{(\alpha)} \text{ for all } R_{(\alpha)} \tag{4.11}$$

$$\Rightarrow \left.\frac{\partial H(\ .\ )}{\partial \hat{R}^*_{(\alpha)}}\right|_{(R_{(\alpha)}, e^T)} = 1 \tag{4.12}$$

and also satisfies certain conditions similar to those given for the class of estimators $G_{(\alpha)}$ at (3.2) and e^T denote the row vector of p unit elements.

It can be shown that

$$E\left(H_{(\alpha)}\right) = R_{(\alpha)} + o\left(n^{-1}\right),$$

and to the first degree of approximation the MSE of $H_{(\alpha)}$ is given by

$$\begin{aligned} \text{MSE}\left(H_{(\alpha)}\right) &= \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) + 2R_{(\alpha)} a^T_{(\alpha)} H^{(1)}\left(R_{(\alpha)}, e^T\right) \\ &\quad + \left(H^{(1)}\left(R_{(\alpha)}, e^T\right)\right)^T E\left(H^{(1)}\left(R_{(\alpha)}, e^T\right)\right) \end{aligned} \tag{4.14}$$

which is minimized for

$$H^{(1)}\left(R_{(\alpha)}, e^T\right) = -R_{(\alpha)} E^{-1} a_{(\alpha)},$$

and thus the resulting minimum mean squared error

$$\text{min.MSE}\left(H_{(\alpha)}\right) = \text{MSE}\left(\hat{R}^{*}_{(\alpha)}\right) - R^{2}_{(\alpha)} a^{T}_{(\alpha)} E^{-1} a_{(\alpha)} \tag{4.14}$$

$$= \text{MSE}\left(\hat{R}^{*}_{(\alpha)}\right) - R^{2}_{(\alpha)} \left[\left\{ \left(\tfrac{1-f}{n}\right) q_{(\alpha)} + \tfrac{W_2(k-1)}{n} q^{(2)}_{(\alpha)} \right\}^{T} + E^{-1} \left\{ \left(\tfrac{1-f}{n}\right) q_{(\alpha)} + \tfrac{W_2(k-1)}{n} q^{(2)}_{(\alpha)} \right\} \right],$$

where $H^{(1)}\left(R_{(\alpha)}, e^{T}\right)$ denote the p elements column vector of first partial derivatives of $H\,(\,.\,)$.

Thus state the following theorem:

THEOREM 4.1 Up to terms of order n^{-1},

$$\text{MSE}\left(H_{(\alpha)}\right) \geq \left[\text{MSE}\left(\hat{R}^{*}_{(\alpha)}\right) - R^{2}_{(\alpha)} a^{T}_{(\alpha)} E^{-1} a_{(\alpha)}\right],$$

with equality holding if

$$H^{(1)}\left(R_{(\alpha)}, e^{T}\right) = -R_{(\alpha)} E^{-1} a_{(\alpha)}.$$

REMARK 4.2 Population means $\overline{X}_1,\ \overline{X}_2,\ ...,\ \overline{X}_p$ are known, incomplete information on the study variates (y_0, y_1) and complete information on the auxiliary variates $x_j\ (j = 1, 2, ..., p)$. In this case observe that n_1 units respond on the study variates (y_0, y_1) but there is complete information on the auxiliary variate $x_j\ (j = 1, 2, ..., p)$ and the population means $\overline{X}_1,\ \overline{X}_2,\ ...,\ \overline{X}_p$ are known. In such a situation define a class of estimators for population parameter $R_{(\alpha)}$ as

$$F_{(\alpha)} = F\left(\hat{R}^{*}_{(\alpha)},\ w^{T}\right) \tag{4.15}$$

where w denotes the column vector of p elements $w_1,\ w_2, ..., w_p$ with $w_j = \overline{x}^{*}_{j}/\overline{X}_j$, $j = 1, 2, ..., p$; $F\left(\hat{R}^{*}_{(\alpha)},\ w^{T}\right)$ is a function of $\left(\hat{R}^{*}_{(\alpha)},\ w^{T}\right)$ such that

$$F\left(R_{(\alpha)}, e^{T}\right) = R_{(\alpha)} \text{ for all } R_{(\alpha)}, \tag{4.16}$$

$$\Rightarrow \left.\frac{\partial F\,(\,.\,)}{\partial \hat{R}^{*}_{(\alpha)}}\right|_{\left(R_{(\alpha)}, e^{T}\right)} = 1 \tag{4.17}$$

and also satisfies certain conditions similar to those given for the class of estimators $G_{(\alpha)}$ at (3.2) and e^{T} denote the row vector of p unit elements.

It can be shown that

$$E\left(F_{(\alpha)}\right) = R_{(\alpha)} + o\left(n^{-1}\right),$$

and to the first degree of approximation, the MSE of $F_{(\alpha)}$ is given by

$$\begin{aligned}\text{MSE}\left(F_{(\alpha)}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) + 2R_{(\alpha)}C^T_{(\alpha)}F^{(1)}\left(R_{(\alpha)}, e^T\right) \\ + \left(F^{(1)}\left(R_{(\alpha)}, e^T\right)\right)^T F\left(F^{(1)}\left(R_{(\alpha)}, e^T\right)\right)\end{aligned} \tag{4.18}$$

where $F^{(1)}\left(R_{(\alpha)}, e^T\right)$ is the p elements column vector of the partial derivatives of $F(\,.\,)$, $C_{(\alpha)} = \left(C_{(\alpha)1},\ C_{(\alpha)2},\ ..., C_{(\alpha)p}\right)$, $C_{(\alpha)j} = \left(\frac{1-f}{n}\right) q_{(\alpha)j}$, $q_{(\alpha)j} = C_{x_j}\left(\rho_{y_0x_j}C_{y_0} - \alpha\rho_{y_1x_j}C_{x_j}\right)$, $F = \left(f_{\text{jl}}\right)_{p\times p}$, $f_{\text{jl}} = \left(\frac{1-f}{n}\right)\rho_{x_jx_l}C_{x_j}C_{x_l}$.

The MSE of $F_{(\alpha)}$ at (4.18) is minimized for

$$F^{(1)}\left(R_{(\alpha)}, e^T\right) = -R_{(\alpha)}F^{-1}C_{(\alpha)} \tag{4.19}$$

and thus the resulting minimum mean squared error of $F_{(\alpha)}$ is given by

$$\begin{aligned}\text{min.MSE}\left(F_{(\alpha)}\right) &= \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}C^T_{(\alpha)}F^{-1}C_{(\alpha)} \\ &= \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}\left(\frac{1-f}{n}\right)q^T_{(\alpha)}F_0^{-1}q_{(\alpha)}\end{aligned} \tag{4.20}$$

where $F_0 = \left(a_{\text{jl}}\right)_{p\times p}$ and $a_{\text{jl}} = \rho_{x_jx_l}C_{x_j}C_{x_l}$.

Thus the following theorem holds.

Theorem 4.2 Up to terms of order n^{-1},

$$\text{MSE}\left(F_{(\alpha)}\right) \geq \left[\text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}C^T_{(\alpha)}F^{-1}C_{(\alpha)}\right],$$

with equality holding if

$$F^{(1)}\left(R_{(\alpha)}, e^T\right) = -R_{(\alpha)}F^{-1}C_{(\alpha)}.$$

Remark 4.3 Population means of auxiliary characters are unknown, incomplete information on the study variates (y_0, y_1) and complete information on the auxiliary variates $x_j\,(j = 1, 2, ..., p)$.

In this case, I use information on $(n_1 + m)$ responding units on the study variates (y_0, y_1) and complete information on the auxiliary variate $x_j\,(j = 1, 2, ..., p)$. Here in formulation of the estimator, in addition to $\overline{x}_j\,(j = 1, 2, ..., p)$ I also use the information on $\overline{x}^*_j\,(j = 1, 2, ..., p)$ which can be easily computed while computing $\overline{y}^*_i\ (i = 0, 1)$. The population means $\overline{x}_j\,(j = 1, 2, ..., p)$ of the auxiliary characters $x_j\,(j = 1, 2, ..., p)$ are not known. With this background define a class of estimators for the parameter $R_{(\alpha)}$ as

$$J_{(\alpha)} = J\left(\hat{R}^*_{(\alpha)},\ z^T\right) \tag{4.21}$$

where z denotes the column vector of p elements $z_1,\ z_2, ..., z_p$, super fix T over a column vector denotes the corresponding row vector, $z_j = \overline{x}_j^*/\overline{x}_j,\ j = 1, 2, ..., p$; $J\left(\hat{R}^*_{(\alpha)},\ z^T\right)$ is a function of $\left(\hat{R}^*_{(\alpha)},\ z^T\right)$ such that

$$J\left(R_{(\alpha)}, e^T\right) = R_{(\alpha)} \text{ for all } R_{(\alpha)} \tag{4.22}$$

$$\Rightarrow \left.\frac{\partial J(\ .\ )}{\partial \hat{R}^*_{(\alpha)}}\right|_{(R_{(\alpha)}, e^T)} = 1 \tag{4.23}$$

and also satisfies certain conditions similar to those given for the class of estimators $G_{(\alpha)}$ at (3.2) and e^T denote the row vector of p unit elements. It can be shown that

$$E\left(J_{(\alpha)}\right) = R_{(\alpha)} + o\left(n^{-1}\right),$$

and to the first degree of approximation the MSE of $J_{(\alpha)}$ is given by

$$\begin{aligned} \text{MSE}\left(J_{(\alpha)}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - 2R_{(\alpha)}\left(a^{(2)}_{(\alpha)}\right)^T J^{(1)}\left(R_{(\alpha)}, e^T\right) \\ + R^2_{(\alpha)}\left(J^{(1)}\left(R_{(\alpha)}, e^T\right)\right)^T M\left(J^{(1)}\left(R_{(\alpha)}, e^T\right)\right) \end{aligned} \tag{4.24}$$

where $J^{(1)}_{(\alpha)}\left(R_{(\alpha)}, e^T\right)$ denote the p elements column vector of the first partial derivatives of $J\left(\hat{R}^*_{(\alpha)},\ z^T\right)$ with respect to $\hat{R}^*_{(\alpha)}$ about the point $\left(R_{(\alpha)}, e^T\right)$;

$\left(a^{(2)}_{(\alpha)}\right)^T = \left(a^{(2)}_{(\alpha)1},\ a^{(2)}_{(\alpha)2},\ ..., a^{(2)}_{(\alpha)p}\right)$, $M = (m_{\text{jl}})_{p\times p}$, $a^{(2)}_{(\alpha)j} = \frac{W_2(k-1)}{n} q^{(2)}_{(\alpha)j},\ j = 1, 2, ..., p$, $\left(a^{(2)}_{(\alpha)}\right)^T = \frac{W_2(k-1)}{n}\left(q^{(2)}_{(\alpha)}\right)^T$, $q^{(2)}_{(\alpha)j} = C_{x_j(2)}\left(\rho_{y_0 x_j(2)} C_{y_0(2)} - \alpha\rho_{y_1 x_j(2)} C_{y_1(2)}\right)$, $j = 1, 2, ..., p$, $m_{\text{jl}} = \frac{W_2(k-1)}{n}\rho_{x_j x_l(2)} C_{x_j(2)} C_{x_l(2)} = \frac{W_2(k-1)}{n} a^{(2)}_{\text{jl}}$, $(j, l) = 1, 2, ..., p$.

The MSE of $J_{(\alpha)}$ at (4.24) is minimized for

$$J^{(1)}\left(R_{(\alpha)}, e^T\right) = -R_{(\alpha)} M^{-1} a^{(2)}_{(\alpha)} = -R_{(\alpha)} M_0^{-1} q^{(2)}_{(\alpha)} \tag{4.25}$$

where $M_0 = \left(a^{(2)}_{\text{jl}}\right)_{p\times p}$, $a^{(2)}_{\text{jl}} = \rho_{x_j x_l(2)} C_{x_j(2)} C_{x_l(2)}$.

Thus the resulting minimum mean squared error of $J_{(\alpha)}$ is given by

$$\begin{aligned} \text{min.MSE}\left(J_{(\alpha)}\right) &= \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}\left(a^{(2)}_{(\alpha)}\right)^T M^{-1}\left(a^{(2)}_{(\alpha)}\right) \\ &= \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}\frac{W_2(k-1)}{n}\left(q^{(2)}_{(\alpha)}\right)^T M_0^{-1}\left(q^{(2)}_{(\alpha)}\right) \end{aligned} \tag{4.26}$$

Thus we state the following theorem.

THEOREM 4.3 Up to terms of order n^{-1},

$$\mathrm{MSE}\left(J_{(\alpha)}\right) \geq \left[\mathrm{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}\left(a^{(2)}_{(\alpha)}\right)^T M^{-1}\left(a^{(2)}_{(\alpha)}\right)\right]$$

with equality holding if

$$J^{(1)}\left(R_{(\alpha)}, e^T\right) = -R_{(\alpha)} M^{-1} a^{(2)}_{(\alpha)}.$$

5. Efficiency comparisons

Note that

$$b^T_{(\alpha)} D^{-1} b_{(\alpha)} = a^T_{(\alpha)} E^{-1} a_{(\alpha)} + \left(F^T E^{-1} a_{(\alpha)} - C_{(\alpha)}\right)^T A^{-1}\left(F^T E^{-1} a_{(\alpha)} - C_{(\alpha)}\right) \tag{5.1}$$

and

$$b^T_{(\alpha)} D^{-1} b_{(\alpha)} = C^T_{(\alpha)} F^{-1} C_{(\alpha)} + \left(a^{(2)}_{(\alpha)}\right)^T M^{-1}\left(a^{(2)}_{(\alpha)}\right) \tag{5.2}$$

where $A = \left(F - F^T E^{-1} F\right)$.

Thus from (3.10), the result follows

$$\begin{aligned}\min.\mathrm{MSE}\left(G_{(\alpha)}\right) = \mathrm{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}\Big[&a^T_{(\alpha)} E^{-1} a_{(\alpha)} \\ &+ \left(F^T E^{-1} a_{(\alpha)} - C_{(\alpha)}\right)^T A^{-1}\left(F^T E^{-1} a_{(\alpha)} - C_{(\alpha)}\right)\Big]\end{aligned} \tag{5.3}$$

$$= \mathrm{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)}\left[C^T_{(\alpha)} F^{-1} C_{(\alpha)} + \left(a^{(2)}_{(\alpha)}\right)^T M^{-1}\left(a^{(2)}_{(\alpha)}\right)\right] \tag{5.4}$$

From (4.23), (4.37), (4.52) and (5.4), one obtain

$$\min.\mathrm{MSE}\left(H_{(\alpha)}\right) - \min.\mathrm{MSE}\left(G_{(\alpha)}\right) =$$

$$R^2_{(\alpha)}\left(F^T E^{-1} a_{(\alpha)} - C_{(\alpha)}\right)^T A^{-1}\left(F^T E^{-1} a_{(\alpha)} - C_{(\alpha)}\right) \geq 0 \tag{5.5}$$

$$\min.\mathrm{MSE}\left(F_{(\alpha)}\right) - \min.\mathrm{MSE}\left(G_{(\alpha)}\right) = R^2_{(\alpha)}\left(a^{(2)}_{(\alpha)}\right)^T M^{-1}\left(a^{(2)}_{(\alpha)}\right) \geq 0 \tag{5.6}$$

$$\min.\mathrm{MSE}\left(J_{(\alpha)}\right) - \min.\mathrm{MSE}\left(G_{(\alpha)}\right) = R^2_{(\alpha)} C^T_{(\alpha)} F^{-1} C_{(\alpha)} \geq 0 \tag{5.7}$$

Thus from (5.5), (5.6) and (5.7), the following inequalities holds

$$\min.\mathrm{MSE}\left(G_{(\alpha)}\right) \leq \min.\mathrm{MSE}\left(H_{(\alpha)}\right) \tag{5.8}$$

$$\min.\mathrm{MSE}\left(G_{(\alpha)}\right) \leq \min.\mathrm{MSE}\left(F_{(\alpha)}\right) \tag{5.9}$$

$$\min.\mathrm{MSE}\left(G_{(\alpha)}\right) \leq \min.\mathrm{MSE}\left(J_{(\alpha)}\right) \tag{5.10}$$

From (5.8), (5.9) and (5.10) it follows that the proposed class of estimators $G_{(\alpha)}$ given by (3.2) is the best (in the sense of having least minimum MSE) among the classes of estimators $G_{(\alpha)}$, $H_{(\alpha)}$, $F_{(\alpha)}$ and $J_{(\alpha)}$.

6. Empirical study

To demonstrate the performance of the suggested estimator relative to usual estimator $\hat{R}^*_{(\alpha)}$ with $\alpha = 1$, consider a natural population data earlier considered by Khare and Sinha (2007). The description of the population is given below:

The data on the physical growth of upper-socio- economic group of 95 school going children of Varanasi under an ICMR study, Department of Pediatrics, BHU during 1983-84 has been taken under study. The first 25% (i.e. 24 children) units have been considered as non-response units. Denote by
y_0: Height (in cm) of the children, y_1: Weight (in kg) of the children,
x_1: Skull circumference (in cm) of the children, x_2: Chest circumference (in cm) of the children.

The required values of the parameters are:

$\overline{Y}_0 = 115.9526$, $\overline{Y}_1 = 19.4968$, $\overline{X}_1 = 51.1726$, $\overline{X}_2 = 55.8611$, $C_{y_0} = 0.0515$,
$C_{y_1} = 0.15613$, $C_{x_1} = 0.03006$, $C_{x_2} = 0.05860$, $C_{y_0(2)} = 0.044$, $C_{y_1(2)} = 0.121$,
$C_{x_1(2)} = 0.02478$, $C_{x_2(2)} = 0.054$, $\rho_{y_0x_1} = 0.374$ $\rho_{y_0x_2} = 0.620$, $\rho_{y_1x_1} = 0.328$,
$\rho_{y_1x_2} = 0.846$, $\rho_{y_0x_1(2)} = 0.571$, $\rho_{y_0x_2(2)} = 0.401$, $\rho_{y_1x_1(2)} = 0.477$, $\rho_{x_1x_1} = 0.297$,
$\rho_{x_1x_1(2)} = 0.570$, $\rho_{y_0y_1} = 0.713$, $\rho_{y_0y_1(2)} = 0.678$

To illustrate results, consider the difference type estimator using two auxiliary variables:

$$t_d = \hat{R}^*_{(\alpha)} + \alpha_1 \left(u_1^* - 1\right) + \alpha_2 \left(u_2^* - 1\right) + \varphi_1 \left(u_1 - 1\right) + \varphi_2 \left(u_2 - 1\right) \tag{6.1}$$

where $\alpha_i' s$ and $\varphi_i' s$, $(i = 1, 2)$ are suitably chosen constants, $u_i^* = \left(\overline{x}_i^* / \overline{X}_i\right)$ and $u_i = \left(\overline{x}_i / \overline{X}_i\right)$, $(i = 1, 2)$.

For the sake of convenience, the MSE of t_d to the first degree of approximation is given by

$$\mathrm{MSE}\left(t_d\right) = \mathrm{MSE}\left(\hat{R}^*_{(\alpha)}\right) + \sum_{j=1}^{2} \alpha_j^2 e_j + 2\alpha_1\alpha_2 e_{12} + 2R_{(\alpha)} \sum_{j=1}^{2} \alpha_j a_{(\alpha)j} + \left(\tfrac{1-f}{n}\right) \left[\sum_{j=1}^{2} \varphi_j^2 C_{x_j}^2 + 2\varphi_1\varphi_2 a_{12}\right.$$

$$\left. + 2R_{(\alpha)} \sum_{j=1}^{2} \varphi_j q_{(\alpha)j} + 2\left\{\alpha_1\varphi_1 C_{x_1}^2 + \alpha_2\varphi_1 a_{12} + \alpha_1\varphi_2 a_{12} + \alpha_2\varphi_2 C_{x_2}^2\right\}\right] \tag{6.2}$$

where

$e_j = \left\{\left(\frac{1-f}{n}\right) C_{x_j}^2 + \frac{W_2(k-1)}{n} C_{x_j(2)}^2\right\}$, $j = 1, 2$; $e_{12} = \left\{\left(\frac{1-f}{n}\right) a_{12} + \frac{W_2(k-1)}{n} a_{12}^{(2)}\right\}$;
$q_{(\alpha)j} = C_{x_j} \left(\rho_{y_0x_j} C_{y_0} - \alpha \rho_{y_1x_j} C_{y_1}\right)$, $j = 1, 2$; $a_{12} = \rho_{x_1x_2} C_{x_1} C_{x_2}$; $a_{12}^{(2)} = \rho_{x_1x_2(2)} C_{x_1(2)} C_{x_2(2)}$;
$q_{(\alpha)j(2)} = C_{x_j(2)} \left(\rho_{y_0x_j(2)} C_{y_0(2)} - \alpha \rho_{y_1x_j(2)} C_{y_1(2)}\right)$, $j = 1, 2$;
$a_{(\alpha)j} = \left\{\left(\frac{1-f}{n}\right) q_{(\alpha)j} + \frac{W_2(k-1)}{n} q_{(\alpha)j(2)}\right\}$, $j = 1, 2$.

Expression (6.2) can also be obtained from (3.8) just by putting the suitable values of the derivatives. The MSE at (6.2) is minimized for

$$\alpha_{10} = R_{(\alpha)} d^*_{(2)} \tag{6.3}$$

$$\alpha_{20} = R_{(\alpha)} d^*_{1(2)} \tag{6.4}$$

$$\varphi_{10} = R_{(\alpha)} \left(d^* - d^*_{(2)} \right) \tag{6.5}$$

$$\varphi_{20} = R_{(\alpha)} \left(d^*_1 - d^*_{1(2)} \right) \tag{6.6}$$

where
$d^* = \frac{\left[q_{(\alpha)2}\rho_{x_1x_2}C_{x_1} - q_{(\alpha)1}C_{x_2}\right]}{\left[C^2_{x_1}C_{x_2}\left(1-\rho^2_{x_1x_2}\right)\right]}$, $d^*_1 = \frac{\left[q_{(\alpha)1}\rho_{x_1x_2}C_{x_2} - q_{(\alpha)2}C_{x_1}\right]}{\left[C_{x_1}C^2_{x_2}\left(1-\rho^2_{x_1x_2}\right)\right]}$,
$d^*_{(2)} = \frac{\left[q_{(\alpha)2(2)}\rho_{x_1x_2(2)}C_{x_1(2)} - q_{(\alpha)1(2)}C_{x_2(2)}\right]}{\left[C^2_{x_1(2)}C_{x_2(2)}\left(1-\rho^2_{x_1x_2(2)}\right)\right]}$, $d^*_{1(2)} = \frac{\left[q_{(\alpha)1(2)}\rho_{x_1x_2(2)}C_{x_2(2)} - q_{(\alpha)2(2)}C_{x_1(2)}\right]}{\left[C_{x_1(2)}C^2_{x_2(2)}\left(1-\rho^2_{x_1x_2(2)}\right)\right]}$.

Putting (6.2)-(6.6) in (6.1), we get the asymptotic optimum estimator (AOE) in the class of estimators t_d as

$$t^{(0)}_d = \hat{R}^*_{(\alpha)} + R_{(\alpha)} \left[d^*_{(2)} \left(u^*_1 - u_1\right) + d^*_{1(2)} \left(u^*_2 - u_2\right) + d^* \left(u_1 - 1\right) + d^*_1 \left(u_2 - 1\right) \right] \tag{6.7}$$

The MSE of $t^{(0)}_d$ to the first degree of approximation is given by

$$\begin{aligned} \mathrm{MSE}\left(t^{(0)}_d\right) = \mathrm{MSE}\left(\hat{R}^*_{(\alpha)}\right) - R^2_{(\alpha)} &\left[\left(\frac{1-f}{n}\right) \frac{\left\{ \left(a_1 C_{x_1} - a_2 C_{x_2}\right)^2 + 2a_1a_2C_{x_1}C_{x_2}\left(1-\rho_{x_1x_2}\right) \right\}}{\left(1-\rho^2_{x_1x_2}\right)} \right. \\ &\left. + \frac{W_2(k-1)}{2} \frac{\left\{ \left(a_{1(2)}C_{x_1(2)} - a_{2(2)}C_{x_2(2)}\right)^2 + 2a_{1(2)}a_{2(2)}C_{x_1(2)}C_{x_2(2)}\left(1-\rho_{x_1x_2(2)}\right) \right\}}{\left(1-\rho^2_{x_1x_2(2)}\right)} \right] \\ &= \mathrm{min.MSE}\left(t_d\right) \end{aligned} \tag{6.8}$$

where
$a_1 = \rho_{y_0x_1}\frac{C_{y_0}}{C_{x_1}} - \rho_{y_1x_1}\frac{C_{y_1}}{C_{x_1}}$, $a_2 = \rho_{y_0x_2}\frac{C_{y_0}}{C_{x_2}} - \rho_{y_1x_2}\frac{C_{y_1}}{C_{x_2}}$, $a_{1(1)} = \rho_{y_0x_1(1)}\frac{C_{y_0(2)}}{C_{x_1(2)}} - \rho_{y_1x_1(2)}\frac{C_{y_1(2)}}{C_{x_1(2)}}$,
$a_{2(2)} = \rho_{y_0x_2(2)}\frac{C_{y_0(2)}}{C_{x_2(2)}} - \rho_{y_1x_2(2)}\frac{C_{y_1(2)}}{C_{x_2(2)}}$.

In practice the optimum values of α_1, α_2, φ_1 and φ_2 given by (6.2)-(6.6) are not known. In such a case it is worth advisable to replace them by their consistent estimators in (6.8) and thus one get an estimator based on "estimated optimum values" as

$$\hat{t}^{(0)}_d = \hat{R}^*_{(\alpha)} \left[1 + \hat{d}^*_{(2)} \left(u^*_1 - u_1\right) + \hat{d}^*_{1(2)} \left(u^*_2 - u_2\right) + \hat{d}^* \left(u_1 - 1\right) + \hat{d}^*_1 \left(u_2 - 1\right) \right] \tag{6.9}$$

where $\hat{d}^*$, $\hat{d}^*_1$, $\hat{d}^*_{(2)}$ and $\hat{d}^*_{1(2)}$ are the consistent estimators of d^*, d^*_1, $d^*_{(2)}$ and $d^*_{1(2)}$ based on the available data under the given sampling design. It can be easily shown to the first degree of approximation that

$$\mathrm{MSE}\left(\hat{t}^{(0)}_d\right) = \mathrm{min.MSE}\left(t_d\right) = \mathrm{min.MSE}\left(t_d\right) \tag{6.10}$$

where MSE $\left(t_d^{(0)}\right)$ isgiven by (6.8).

Further consider the following difference type estimators:

$$t_{d1} = \hat{R}^*_{(\alpha)} + \alpha_1 \left(u_1^* - 1\right) + \alpha_2 \left(u_2^* - 1\right) \tag{6.11}$$

$$t_{d2} = \hat{R}^*_{(\alpha)} + \varphi_1 \left(u_1 - 1\right) + \varphi_2 \left(u_2 - 1\right) \tag{6.12}$$

$$t_{d3} = \hat{R}^*_{(\alpha)} + \lambda_1 \left(z_1 - 1\right) + \lambda_2 \left(z_2 - 1\right) \tag{6.13}$$

where $z_1 = \overline{x}_1^*/\overline{x}_1,\ z_2 = \overline{x}_2^*/\overline{x}_2,\ \alpha_i's,\ \varphi_i's$ and $\lambda_{i's},\ (i = 1, 2)$ are suitably chosen constants. To the first degree of approximation, the minimum MSE of t_{d1}, t_{d2} and t_{d3} are respectively given by

$$\text{min.MSE}\left(t_{d1}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - \frac{R^2_{(\alpha)}}{\left(e_1 e_2 - e_{12}^2\right)} \left[\left(a_{(\alpha)1}\right)^2 e_2 + \left(a_{(\alpha)2}\right)^2 e_1 - 2a_{(\alpha)1} a_{(\alpha)2} e_{12}\right] \tag{6.14}$$

for optimum values of α_1 and α_2 given by

$$\left.\begin{array}{l} \alpha_{10}^* = \frac{R_{(\alpha)}\left[a_{(\alpha)2}e_{12} - a_{(\alpha)1}e_2\right]}{\left(e_1e_2 - e_{12}^2\right)} \\ \alpha_{20}^* = \frac{R_{(\alpha)}\left[a_{(\alpha)1}e_{12} - a_{(\alpha)2}e_1\right]}{\left(e_1e_2 - e_{12}^2\right)} \end{array}\right\}, \tag{6.15}$$

$$\begin{aligned} \text{min.MSE}\left(t_{d2}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - \left(\tfrac{1-f}{n}\right) \frac{R^2_{(\alpha)}}{\left[C^2_{x_1(2)} C^2_{x_2(2)} - \left(a_{12}^{(2)}\right)^2\right]} &\left[\left(q_{(\alpha)1(2)}\right)^2 C^2_{x_2(2)}\right. \\ &\left. + \left(q_{(\alpha)2(2)}\right)^2 C^2_{x_1(2)} - 2q_{(\alpha)1(2)} q_{(\alpha)2(2)} a_{12}^{(2)}\right], \end{aligned} \tag{6.16}$$

for optimum values of φ_1 and φ_2 given by

$$\left.\begin{array}{l} \varphi_{10}^* = \frac{R_{(\alpha)}\left[q_{(\alpha)2(2)} a_{12}^{(2)} - q_{(\alpha)1(2)} C^2_{x_2(2)}\right]}{\left[C^2_{x_1(2)} C^2_{x_2(2)} - \left(a_{12}^{(2)}\right)^2\right]} \\ \varphi_{20}^* = \frac{R_{(\alpha)}\left[q_{(\alpha)1(2)} a_{12}^{(2)} - q_{(\alpha)2(2)} C^2_{x_1(2)}\right]}{\left[C^2_{x_1(2)} C^2_{x_2(2)} - \left(a_{12}^{(2)}\right)^2\right]} \end{array}\right\}, \tag{6.17}$$

$$\begin{aligned} \text{min.MSE}\left(t_{d3}\right) = \text{MSE}\left(\hat{R}^*_{(\alpha)}\right) - \frac{W_2(k-1)}{n} \frac{R^2_{(\alpha)}}{\left[C^2_{x_1(2)} C^2_{x_2(2)} - \left(a_{12}^{(2)}\right)^2\right]} &\left[\left(q_{(\alpha)1(2)}\right)^2 C^2_{x_2(2)}\right. \\ &\left. + \left(q_{(\alpha)2(2)}\right)^2 C^2_{x_1(2)} - 2q_{(\alpha)1(2)} q_{(\alpha)2(2)} a_{12}^{(2)}\right], \end{aligned} \tag{6.18}$$

for optimum values of λ_1 and λ_2 given by

$$\left.\begin{array}{l} \lambda_{10} = \frac{R_{(\alpha)}\left[q_{(\alpha)2(2)} a_{12}^{(2)} - q_{(\alpha)1(2)} C^2_{x_2(2)}\right]}{\left[C^2_{x_1(2)} C^2_{x_2(2)} - \left(a_{12}^{(2)}\right)^2\right]} \\ \lambda_{20} = \frac{R_{(\alpha)}\left[q_{(\alpha)1(2)} a_{12}^{(2)} - q_{(\alpha)2(2)} C^2_{x_1(2)}\right]}{\left[C^2_{x_1(2)} C^2_{x_2(2)} - \left(a_{12}^{(2)}\right)^2\right]} \end{array}\right\}. \tag{6.19}$$

Estimators based on estimated values of $(\alpha_{10}^*,\ \alpha_{20}^*)$, $(\varphi_{10}^*,\ \varphi_{20}^*)$ and $(\lambda_{10},\ \lambda_{20})$ are respectively given by

$$\hat{t}_{d1}^{(0)} = \hat{R}_{(\alpha)}^{*} + \hat{\alpha}_{10}^{*}\left(u_1^{*} - 1\right) + \hat{\alpha}_{20}^{*}\left(u_2^{*} - 1\right) \tag{6.20}$$

$$\hat{t}_{d2}^{(0)} = \hat{R}_{(\alpha)}^{*} + \hat{\varphi}_{10}^{*}\left(u_1 - 1\right) + \hat{\varphi}_{20}^{*}\left(u_2 - 1\right) \tag{6.21}$$

$$\hat{t}_{d3}^{(0)} = \hat{R}_{(\alpha)}^{*} + \hat{\lambda}_{10}\left(z_1 - 1\right) + \hat{\lambda}_{20}\left(z_2 - 1\right) \tag{6.22}$$

where $\hat{\alpha}_{10}^{*}$, $\hat{\alpha}_{20}^{*}$, $\hat{\varphi}_{10}^{*}$, $\hat{\varphi}_{20}^{*}$, $\hat{\lambda}_{10}$ and $\hat{\lambda}_{20}$ are the consistent estimates of the optimum value α_{10}^{*}, α_{20}^{*}, φ_{10}^{*}, φ_{20}^{*}, λ_{10} and λ_{20} respectively based on the data available under the given sampling design. To the first degree of approximation, it can be shown that

$$\text{MSE}\left(\hat{t}_{d1}^{(0)}\right) = \text{min.MSE}\left(t_{d1}\right) \tag{6.23}$$

$$\text{MSE}\left(\hat{t}_{d2}^{(0)}\right) = \text{min.MSE}\left(t_{d2}\right) \tag{6.24}$$

$$\text{MSE}\left(\hat{t}_{d3}^{(0)}\right) = \text{min.MSE}\left(t_{d3}\right) \tag{6.25}$$

where $\text{min.MSE}\left(t_{d1}\right)$, $\text{min.MSE}\left(t_{d2}\right)$ and $\text{min.MSE}\left(t_{d3}\right)$ are respectively given by (6.14), (6.16) and (6.18).

I have computed the percent relative efficiencies (PREs) of $t_d^{(0)}\left(\text{or } \hat{t}_d^{(0)}\right)$, $t_{d1}^{(0)}\left(\text{or } \hat{t}_{d1}^{(0)}\right)$, $t_{d2}^{(0)}\left(\text{or } \hat{t}_{d2}^{(0)}\right)$ and $t_{d3}^{(0)}\left(\text{or } \hat{t}_{d3}^{(0)}\right)$ with respect to usual estimator $\hat{R}_{(\alpha)}^{*}$ with $\alpha = 1$ where $t_{d1}^{(0)}$, $t_{d2}^{(0)}$ and $t_{d3}^{(0)}$ are respectively the optimum estimators in t_{d1}, t_{d2} and t_{d3}.

The findings are given in Table 1.

Table 1. Percent relative efficiencies of the estimators with respect to $\hat{R}_{(\alpha)}^{*}$ with $\alpha = 1$ for fixed n and different values of k.

Estimator	(1/k)			
	(1/5)	(1/4)	(1/3)	(1/2)
$\hat{R}^{*}$	100.00	100.00	100.00	100.00
$t_d^{(0)}\left(or\ \hat{t}_d^{(0)}\right)$	368.22	352.24	332.44	309.50
$t_{d1}^{(0)}\left(or\ \hat{t}_{d1}^{(0)}\right)$	240.12	248.64	261.16	282.91
$t_{d2}^{(0)}\left(or\ \hat{t}_{d2}^{(0)}\right)$	147.86	158.73	175.89	207.44
$t_{d3}^{(0)}\left(or\ \hat{t}_{d3}^{(0)}\right)$	117.68	114.78	111.13	106.39

It is observed from Table 1 that the percent relative efficiencies of $t_d^{(0)}\left(\text{or } \hat{t}_d^{(0)}\right)$ and $t_{d3}^{(0)}\left(\text{or } \hat{t}_{d3}^{(0)}\right)$ decrease while the percent relative efficiencies of $t_{d1}^{(0)}\left(\text{or } \hat{t}_{d1}^{(0)}\right)$ and $t_{d2}^{(0)}\left(\text{or } \hat{t}_{d2}^{(0)}\right)$ increase with respect to $\hat{R}^{*}$ as the sub-sampling fraction increases. It has also been perceived that $t_d^{(0)}\left(\text{or } \hat{t}_d^{(0)}\right)$ is the best among $\hat{R}^{*}$, $t_{d1}^{(0)}\left(\text{or } \hat{t}_{d1}^{(0)}\right)$, $t_{d2}^{(0)}\left(\text{or } \hat{t}_{d2}^{(0)}\right)$ and $t_{d3}^{(0)}\left(\text{or } \hat{t}_{d3}^{(0)}\right)$. Thus, the suggested estimator $t_d^{(0)}\left(\text{or } \hat{t}_d^{(0)}\right)$ is to be preferred for its use in practice, when the difference type estimator using two auxiliary variables is used.

7. Conclusion

In the present problem, some classes of estimators for ratio, product and mean are discussed by using multi auxiliary in different situations in the presence of non-response and their properties have been studied. Conditions for attaining minimum mean squared error of the proposed classes of estimators have also been obtained. Estimators based on estimated optimum values have been obtained with their approximate mean squared error. Due to the non-availability of the data, I have tried to show the performance of the suggested estimator relative to usual estimator $\hat{R}^*_{(\alpha)}$ with $\alpha = 1$ for two auxiliary variables. The performance of the suggested estimator is preferable when the non-response occurs on the study as well as auxiliary variables.

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Appendix I

Putting $\alpha = 1,\ -1,\ 0$ in (3.2) we get

(i) a class of estimators for ratio R as

$$G_{(1)} = G\left(\hat{R}^*,\ u^T\right) \tag{I-1}$$

(ii) a class of estimators for product P as

$$G_{(-1)} = G\left(\hat{P}^*,\ u^T\right) \tag{I-2}$$

(iii) a class of estimators for population mean $\overline{Y}_0$ as

$$G_{(0)} = G\left(\overline{y}_0^*,\ u^T\right) \tag{I-3}$$

The minimum mean squared errors of the estimators $G_{(1)},\ G_{(-1)}$ and $G_{(0)}$ are respectively given by

$$\text{min.MSE}\left(G_{(1)}\right) = \text{MSE}\left(\hat{R}^*\right) - R^2 b^T_{(1)} D^{-1} b_{(1)} \tag{I-4}$$

$$\text{min.MSE}\left(G_{(-1)}\right) = \text{MSE}\left(\hat{P}^*\right) - P^2 b^T_{(-1)} D^{-1} b_{(-1)} \tag{I-5}$$

$$\text{min.MSE}\left(G_{(0)}\right) = \text{Var}\left(\overline{y}_0^*\right) - \overline{Y}^2_{(0)} b^T_{(0)} D^{-1} b_{(0)} \tag{I-6}$$

where

$$\begin{aligned}\text{MSE}\left(\hat{R}^*\right) = R^2 \Big[&\left(\tfrac{1-f}{n}\right)\left(C^2_{y_0} + C^2_{y_1} - 2\rho_{y_0y_1} C_{y_0} C_{y_1}\right) \\ &+ \tfrac{W_2(k-1)}{n}\left(C^2_{y_0(2)} + C^2_{y_1(2)} - 2\rho_{y_0y_1(2)} C_{y_0(2)} C_{y_1(2)}\right)\Big],\end{aligned} \tag{I-7}$$

$$\mathrm{MSE}\left(\hat{P}^{*}\right) = P^{2}\left[\left(\tfrac{1-f}{n}\right)\left(C_{y_0}^{2}+C_{y_1}^{2}+2\rho_{y_0y_1}C_{y_0}C_{y_1}\right) + \tfrac{W_2(k-1)}{n}\left(C_{y_0(2)}^{2}+C_{y_1(2)}^{2}+2\rho_{y_0y_1(2)}C_{y_0(2)}C_{y_1(2)}\right)\right], \tag{I-8}$$

are the mean squared errors of $\hat{R}^{*}$ and $\hat{P}^{*}$ to the first degree of approximation, respectively, and

$$\mathrm{Var}\left(\overline{y}_0^{*}\right) = \left[\left(\frac{1-f}{n}\right)S_0^{*^2} + \frac{W_2\,(k-1)}{n}S_{0(2)}^{*^2}\right] \tag{I-9}$$

where

$b_{(1)}^{T} = \left(a_{(1)1},\ a_{(1)2},\ ...,a_{(1)p},\ C_{(1)1},\ C_{(1)2},\ ...,C_{(1)p}\right)$,
$b_{(-1)}^{T} = \left(a_{(-1)1},\ a_{(-1)2},\ ...,a_{(-1)p},\ C_{(-1)1},\ C_{(-1)2},\ ...,C_{(-1)p}\right)$,
$b_{(0)}^{T} = \left(a_{(0)1},\ a_{(0)2},\ ...,a_{(0)p},\ C_{(0)1},\ C_{(0)2},\ ...,C_{(0)p}\right)$,
$a_{(1)j} = \left[\left(\frac{1-f}{n}\right)q_{(1)j} + \frac{W_2(k-1)}{n}q_{(1)j}^{(2)}\right],\ j = 1,\ 2, ...,\ p;$
$a_{(-1)j} = \left[\left(\frac{1-f}{n}\right)q_{(-1)j} + \frac{W_2(k-1)}{n}q_{(-1)j}^{(2)}\right],\ j = 1,\ 2, ...,\ p;$
$a_{(0)j} = \left[\left(\frac{1-f}{n}\right)q_{(0)j} + \frac{W_2(k-1)}{n}q_{(0)j}^{(2)}\right],\ j = 1,\ 2, ...,\ p;$
$q_{(1)j} = C_{x_j}\left(\rho_{y_0x_j}C_{y_0}-\rho_{y_1x_j}C_{y_1}\right),\ q_{(1)j}^{(2)} = C_{x_j(2)}\left(\rho_{y_0x_j(2)}C_{y_0(2)}-\rho_{y_1x_j(2)}C_{y_1(2)}\right)$,
$q_{(-1)j} = C_{x_j}\left(\rho_{y_0x_j}C_{y_0}+\rho_{y_1x_j}C_{y_1}\right),\ q_{(-1)j}^{(2)} = C_{x_j(2)}\left(\rho_{y_0x_j(2)}C_{y_0(2)}+\rho_{y_1x_j(2)}C_{y_1(2)}\right)$,
$q_{(0)j} = \rho_{y_0x_j}C_{y_0}C_{x_j}$, $q_{(0)j}^{(2)} = \rho_{y_0x_j(2)}C_{y_0(2)}C_{x_j(2)}$, $C_{(1)j} = \left(\frac{1-f}{n}\right)q_{(1)j}$,
$C_{(-1)j} = \left(\frac{1-f}{n}\right)q_{(-1)j},\ C_{(0)j} = \left(\frac{1-f}{n}\right)q_{(0)j}$.

8. Appendix II

Putting $\alpha = 1,\ -1,\ 0$ in (4.10) we get the class of estimators

(i) for ratio R as

$$H_{(1)} = H\left(\hat{R}^{*},\ \nu^{T}\right) \tag{II-1}$$

(ii) for product P as

$$H_{(-1)} = H\left(\hat{P}^{*},\ \nu^{T}\right) \tag{II-2}$$

(iii) for population mean $\overline{Y}_0$ as

$$H_{(0)} = H\left(\overline{y}_0^{*},\ \nu^{T}\right) \tag{II-3}$$

The minimum mean squared errors of the estimators $H_{(1)}$, $H_{(-1)}$ and $H_{(0)}$ can be obtained from (4.13) by putting $\alpha = 1,\ -1,\ 0$ and are respectively given by

$$\min.\mathrm{MSE}\left(H_{(1)}\right) = \mathrm{MSE}\left(\hat{R}^{*}\right) - R^{2}a_{(1)}^{T}E^{-1}a_{(1)} \tag{II-4}$$

$$\min.\mathrm{MSE}\left(H_{(-1)}\right) = \mathrm{MSE}\left(\hat{P}^{*}\right) - P^{2}a_{(-1)}^{T}E^{-1}a_{(-1)} \tag{II-5}$$

$$\min.\mathrm{MSE}\left(H_{(0)}\right) = \mathrm{Var}\left(\overline{y}_{0}^{*}\right) - \overline{Y}_{(0)}^{2}a_{(0)}^{T}E^{-1}a_{(0)} \tag{II-6}$$

where
$a_{(1)} = \left(a_{(1)1},\ a_{(1)2},\ ..., a_{(1)p}\right)$,
$a_{(0)} = \left(a_{(0)1},\ a_{(0)2},\ ..., a_{(0)p}\right)$,
$a_{(-1)} = \left(a_{(-1)1},\ a_{(-1)2},\ ..., a_{(-1)p}\right)$,
$a_{(1)j} = \left[\left(\frac{1-f}{n}\right)q_{(1)j} + \frac{W_2(k-1)}{n}q_{(1)}^{(2)}j\right],\ j = 1,\ 2, ...,\ p;$
$a_{(-1)j} = \left[\left(\frac{1-f}{n}\right)q_{(-1)j} + \frac{W_2(k-1)}{n}q_{(-1)}^{(2)}j\right],\ j = 1,\ 2, ...,\ p;$
$a_{(0)j} = \left[\left(\frac{1-f}{n}\right)q_{(0)j} + \frac{W_2(k-1)}{n}q_{(0)}^{(2)}j\right],\ j = 1,\ 2, ...,\ p;$

where
$q_{(1)j},\ q_{(-1)j},\ q_{(0)j},\ q_{(1)j}^{(2)},\ q_{(-1)j}^{(2)},\ q_{(0)j}^{(2)}$ are same as defined earlier.
It is to be mentioned that the class of estimators

$$t_1 = \hat{R}^{*}h\left(\nu^{T}\right) \tag{II-7}$$

of the ratio R reported by Khare and Sinha (2007) is a member of the proposed class of estimator $H_{(1)}$. To the first degree of approximation,

$$\min.\mathrm{MSE}\left(t_1\right) = \min.\mathrm{MSE}\left(H_{(1)}\right) \tag{II-8}$$

where $\min.\mathrm{MSE}\left(H_{(1)}\right)$ is given by (II-2).

9. Appendix III

Putting $\alpha = 1,\ -1,\ 0$ in (4.15) we get the class of estimators

(i) for ratio R as

$$F_{(1)} = F\left(\hat{R}^{*},\ w^{T}\right) \tag{III-1}$$

(ii) for product P as

$$F_{(-1)} = F\left(\hat{P}^{*},\ w^{T}\right) \tag{III-2}$$

(iii) for population mean $\overline{Y}_0$ as

$$F_{(0)} = F\left(\overline{y}_{0}^{*},\ w^{T}\right) \tag{III-3}$$

The minimum mean squared errors of the estimators $F_{(1)},\ F_{(-1)}$ and $F_{(0)}$ are respectively given by

$$\min.\mathrm{MSE}\left(F_{(1)}\right) = \mathrm{MSE}\left(\hat{R}^{*}\right) - R^{2}C_{(1)}^{T}F^{-1}C_{(1)} \tag{III-4}$$

$$\min.\mathrm{MSE}\left(F_{(-1)}\right) = \mathrm{MSE}\left(\hat{P}^{*}\right) - P^{2}C_{(-1)}^{T}F^{-1}C_{(-1)} \tag{III-5}$$

$$\min.\mathrm{MSE}\left(F_{(0)}\right) = \mathrm{Var}\left(\overline{y}_{0}^{*}\right) - \overline{Y}_{(0)}^{2}C_{(0)}^{T}F^{-1}C_{(0)} \tag{III-6}$$

where

$C_{(1)} = \left(C_{(1)1},\ C_{(1)2}, ..., C_{(1)p}\right), C_{(0)} = \left(C_{(0)1},\ C_{(0)2}, ..., C_{(0)p}\right), C_{(-1)} = \left(C_{(-1)1},\ C_{(-1)2}, ..., C_{(-1)p}\right)$
$C_{(1)j} = \left(\frac{1-f}{n}\right) q_{(1)j},\ j = 1,\ 2, ...,\ p,$ $C_{(-1)j} = \left(\frac{1-f}{n}\right) q_{(-1)j},\ j = 1,\ 2, ...,\ p,$
$C_{(0)j} = \left(\frac{1-f}{n}\right) q_{(0)j},\ j = 1,\ 2, ...,\ p,$
$q_{(1)j},\ q_{(-1)j}$ and $q_{(0)j},\ j = 1, 2, ..., p$ are same as defined earlier.

Khare and Sinha (2007) suggested a class of estimators for ratio R as

$$t_{2} = \hat{R}^{*} f\left(w^{T}\right) \tag{III-7}$$

where $f\left(w^{T}\right)$ is a function of $w^{T} = (w_{1},\ w_{2}, ..., w_{p})$ such that $f\left(e^{T}\right) = 1$. The estimator t_2 is due to Khare and Sinha (2007) a member of the class $F_{(1)}$ defined at (III-7). The minimum MSE of t_2 is given by

$$\min.\mathrm{MSE}\left(t_{2}\right) = \min.\mathrm{MSE}\left(F_{(1)}\right) = \mathrm{MSE}\left(\hat{R}^{*}\right) - R^{2}C_{(1)}^{T}F^{-1}C_{(1)} \tag{III-8}$$

Appendix IV

Putting $\alpha = 1,\ -1,\ 0$ in (4.21) we get the class of estimators

(i) for ratio R as

$$J_{(1)} = J\left(\hat{R}^{*},\ z^{T}\right) \tag{IV-1}$$

(ii) for product P as

$$J_{(-1)} = J\left(\hat{P}^{*},\ z^{T}\right) \tag{IV-2}$$

(iii) for population mean $\overline{Y}_0$ as

$$J_{(0)} = J\left(\overline{y}_{0}^{*},\ z^{T}\right) \tag{IV-3}$$

The minimum mean squared errors of the estimators $J_{(1)},\ J_{(-1)}$ and $J_{(0)}$ are respectively given by

$$\text{min.MSE}\left(J_{(1)}\right) = \text{MSE}\left(\hat{R}^*\right) - R^2\left(a_{(1)}^{(2)}\right)^T M^{-1}\left(a_{(1)}^{(2)}\right) \tag{IV-4}$$

$$\text{min.MSE}\left(J_{(-1)}\right) = \text{MSE}\left(\hat{P}^*\right) - P^2\left(a_{(-1)}^{(2)}\right)^T M^{-1}\left(a_{(-1)}^{(2)}\right) \tag{IV-5}$$

$$\text{min.MSE}\left(J_{(0)}\right) = \text{MSE}\left(\bar{y}_0^*\right) - \bar{Y}_0^2\left(a_{(0)}^{(2)}\right)^T M^{-1}\left(a_{(0)}^{(2)}\right) \tag{IV-6}$$

where

$a_{(1)}^{(2)} = \left(a_{(1)1}^{(2)},\ a_{(1)2}^{(2)},\ a_{(1)3}^{(2)}, ..., a_{(1)p}^{(2)}\right)$, $a_{(-1)}^{(2)} = \left(a_{(-1)1}^{(2)},\ a_{(-1)2}^{(2)},\ a_{(-1)3}^{(2)}, ..., a_{(-1)p}^{(2)}\right)$, $a_{(0)}^{(2)} = \left(a_{(0)1}^{(2)},\ a_{(0)2}^{(2)},\ a_{(0)3}^{(2)}, ..., a_{(0)p}^{(2)}\right)$, $a_{(1)j}^{(2)} = \frac{W_2(k-1)}{n} q_{(1)j}^{(2)}$, $a_{(-1)j}^{(2)} = \frac{W_2(k-1)}{n} q_{(-1)j}^{(2)}$, $a_{(0)j}^{(2)} = \frac{W_2(k-1)}{n} q_{(0)j}^{(2)}$, $q_{(1)j}^{(2)} = C_{x_j(2)}\left(\rho_{y_0x_j(2)}C_{y_0} - \rho_{y_1x_j(2)}C_{y_1(2)}\right)$, $q_{(-1)j}^{(2)} = C_{x_j(2)}\left(\rho_{y_0x_j(2)}C_{y_0} + \rho_{y_1x_j(2)}C_{y_1(2)}\right)$, $q_{(0)j}^{(2)} = \rho_{y_0x_j(2)}C_{y_0}C_{x_j(2)}$, $j = 1, 2, ..., p$.

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