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Identifiability problems in some non-Gaussian spatial random fields

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Abstract

The multivariate skew-normal distribution and the elliptically contoured distributions have been developed to model a sample of independent and identically distributed random vectors. Recently, various proposals have been made to extend those distributions to the setting of spatial random fields. We describe identifiability problems associated with inference for those proposals and suggest simple remedies. We also describe some properties of the resulting spatial random fields.

Keywords: Covariance · Elliptically contoured · Identifiability · Skew-elliptical · Skew-normal · Skew- t .

Mathematics Subject Classification: Primary 62M30 · Secondary 62M40.

1. INTRODUCTION

Flexible parametric models for multivariate non-Gaussian distributions have received sustained attention in recent years. For instance, the multivariate skew-normal distribution introduced by Azzalini and Dalla Valle (1996) allows skewness in the data to be modeled and includes the multivariate normal distribution as a special case; see the book edited by Genton (2004) for an overview, further details and extensions. Although this multivariate distribution was originally aimed at modeling a sample of independent and identically distributed random vectors, various proposals have recently emerged to use it to define skew-Gaussian spatial random fields; see, e.g., Kim and Mallick (2003, 2004) and Kim et al. (2004). The spatial situation corresponds to observing a single random vector from the multivariate skew-normal distribution, the dimension of which represents the number of spatial observations. We demonstrate in this article that a spatial random field with a multivariate skew-normal joint distribution as defined by Azzalini and Dalla Valle (1996) cannot be identified correctly with probability 1 by a single realization, even if the number of spatial locations increases to infinity, and we propose a simple remedy.

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Specifically, consider a real-valued spatial random field $\{Y(s): s \in \mathbb{R}^d, d \geq 1\}$, with finite second-order moments. For inferential purposes, e.g., such as likelihood-based procedures, the random field $Y(s)$ is often assumed to be Gaussian. Although this assumption brings mathematical tractability to many spatial statistics problems, it is clearly unrealistic for a large number of practical data analyses. Indeed, the distribution of spatial data often exhibits skewness and heavier tails than the Gaussian distribution. We first consider a skew-Gaussian random field given by

$$Y(s) = \delta|Z_1| + (1 - \delta^2)^{1/2}Z_2(s), \quad (1)$$

where Z_1 is a standard normal random variable, independent of the zero-mean and unit-variance Gaussian random field $Z_2(s)$, and $\delta \in (-1, 1)$. The finite-dimensional distributions of the spatial random field given in Equation (1) are multivariate skew-normal as defined by Azzalini and Dalla Valle (1996). Kim and Mallick (2003, 2004) and Kim et al. (2004) applied such a multivariate skew-normal distribution to some spatial datasets. Although this is an appealing construction, we show in the next section that the random field given in Equation (1) has an identifiability problem.

The spatial random field construction in Equation (1) can be viewed as the sum of a Gaussian random field and a non-Gaussian random variable. This additive structure can be replaced by a multiplicative one with a random scale factor, leading to so-called elliptically contoured multivariate distributions; see Fang et al. (1990). Those distributions are useful for modeling a sample of independent and identically distributed random vectors with heavier or lighter tails than the Gaussian distribution. Ma (2009, 2010a) extended this idea to the construction of elliptically contoured spatial random fields given by

$$Y(s) = (E[R^2])^{-1/2}RZ(s), \quad (2)$$

where $Z(s)$ is a zero-mean Gaussian random field with covariance function $C_Z(s_1, s_2)$, with $s_1, s_2 \in \mathbb{R}^d$, and R is a non-negative random variable with finite second moment and independent of the process $Z(s)$. Ma (2009, 2010a) showed that the spatial random field defined by Equation (2) is also zero-mean with covariance function $C_Y(s_1, s_2) = C_Z(s_1, s_2)$ and has elliptically contoured finite-dimensional distributions. Ma (2009) derived some explicit finite sample distributions of the process given in Equation (2). Although these distributions are useful for modeling multivariate non-Gaussian data when independent and identically distributed multivariate samples are available, we show that they have the same identifiability problem as for Equation (1) when only a single realization from the model given in Equation (2) is available.

The paper is organized as follows. In Section 2, we describe the identifiability problems associated with skew-Gaussian random fields and elliptically contoured random fields. We propose simple remedies in Section 3 and describe some properties of the resulting random fields. We conclude with a discussion in Section 4. The proofs of our results are provided in the Appendix.

2. IDENTIFIABILITY PROBLEMS

2.1 SKEW-GAUSSIAN RANDOM FIELDS

We show that the multivariate skew-normal distribution defined by Azzalini and Dalla Valle (1996) is problematic when it is applied to a single realization of a spatial process.

Recall that a sample Y_1, \dots, Y_n has a multivariate skew-normal distribution if

$$Y_i = \tau_i |Z_0| + \rho_i Z_i, \quad i = 1, \dots, n,$$

where Z_0 and $Z = (Z_1, \dots, Z_n)^\top$ are independent, $Z_0 \sim N(0, 1)$ and $Z \sim N_n(0, \Omega)$, with Ω a correlation matrix, $\tau_i \in \mathbb{R}$ and $\rho_i > 0$. If we assume that the first moments and the second moments of Y_i are all equal, which is an assumption that is commonly made in spatial statistics, we must have $\tau_i = \tau$ and $\rho_i = \rho$, for $i = 1, \dots, n$. Thus, the model parameters are τ , ρ and Ω .

Such an n -variate distribution has been applied to spatial data in an attempt to incorporate skewed marginal distributions; see, e.g., Kim and Mallick (2003, 2004). If n observations $Y(s_1), \dots, Y(s_n)$ have the multivariate skew-normal distribution as defined by Azzalini and Dalla Valle (1996), it can be written as

$$Y(s_i) = \tau |Z_0| + \rho Z(s_i), \quad i = 1, \dots, n, \tag{3}$$

where $Y(s_i)$ denotes the process at location s_i , $Z(s_i)$ is the latent variable at location s_i and Z_0 is a latent variable independent of the $Z(s_i)$'s. Marginally, $Y(s_i)$ defined by Equation (3) has a univariate skew-normal distribution (see Azzalini, 1985) with probability density function given by

$$\frac{2}{\sqrt{\tau^2 + \rho^2}} \phi\left(\frac{y}{\sqrt{\tau^2 + \rho^2}}\right) \Phi\left(\frac{\tau}{\rho\sqrt{\tau^2 + \rho^2}}\right), \quad y \in \mathbb{R}, \tag{4}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density and cumulative distribution functions of a $N(0, 1)$ distribution, respectively.

The finite-dimensional joint probability density function of $Y = (Y(s_1), \dots, Y(s_n))^\top$ is

$$\begin{aligned} f_Y(y) &= 2 \int_{\mathbb{R}_+} \phi_n(y - \tau w \mathbf{1}_n; \rho^2 \Omega) \phi(w) dw \\ &= 2 \phi_n(y; \Psi) \Phi\left(\tau(1 - \tau^2 \mathbf{1}_n^\top \Psi^{-1} \mathbf{1}_n)^{-1/2} \mathbf{1}_n^\top \Psi^{-1} y\right), \end{aligned} \tag{5}$$

where $\phi_n(y; \Omega)$ denotes the probability density function of a $N_n(0, \Omega)$ distribution, $\mathbf{1}_n \in \mathbb{R}^n$ is a vector of ones, and $\Psi = \rho^2 \Omega + \tau^2 \mathbf{1}_n \mathbf{1}_n^\top$. The probability density function in Equation (5) is of Azzalini and Dalla Valle (1996)'s type and reduces to (4) when $n = 1$. From Equation (3), the covariance matrix of the vector Y is given by

$$\text{Var}[Y] = \Psi - \frac{2}{\pi} \tau^2 \mathbf{1}_n \mathbf{1}_n^\top = \left(1 - \frac{2}{\pi}\right) \tau^2 \mathbf{1}_n \mathbf{1}_n^\top + \rho^2 \Omega,$$

which is positive definite for any τ and ρ .

The model given in Equation (3) is problematic for statistical inference. Firstly, the data have no information about the variability of Z_0 and no information about the skewness of the marginal distribution. Secondly, there is an upper bound on the amount of allowable skewness described by τ when $\Psi = I_n$, where I_n denotes the identity matrix. In that case, $\rho^2 \Omega = I_n - \tau^2 \mathbf{1}_n \mathbf{1}_n^\top$ is positive definite if and only if $|\tau| < 1/\sqrt{n}$. Therefore, if the number n of observations is large, then the allowable skewness produced by τ in Equation (3), when $\Psi = I_n$, must be small. Moreover, when $\Psi = I_n$, $Y = \delta |Z_0| + (I_n - \delta \delta^\top)^{1/2} Z_1$, where $\delta = \tau \mathbf{1}_n$, $Z_0 \sim N(0, 1)$ and Z_0 is independent of $Z_1 \sim N_n(0, I_n)$. Thus, Y has the classical Azzalini and Dalla Valle (1996) multivariate skew-normal distribution in that case. If $\Psi \neq I_n$, then there is no restriction on τ .

The next proposition serves as a formal justification of the estimation problem we just discussed. Its proof is provided in the Appendix.

PROPOSITION 2.1 Let Y_i , for $i = 1, 2, \dots$, have equal means and equal variances and have the previously described multivariate skew-normal distribution with parameters τ and ρ . Then, for any two sets of parameters (τ_1, ρ) and (τ_2, ρ) , the two associated probability measures are equivalent on the paths of Y_i , for $i = 1, 2, \dots$.

One direct implication of Proposition 2.1 is that the parameter τ is not consistently estimable. If we let $P_{\tau, \rho}$ denote the probability measure on the σ -algebra $\mathcal{F} = \sigma(Y_i, i = 1, 2, \dots)$, Proposition 2.1 implies that there do not exist estimators $g_n(Y_1, \dots, Y_n)$ such that, for any τ ,

$$P_{\tau, \rho} \left(\lim_{n \rightarrow \infty} g_n(Y_1, \dots, Y_n) \rightarrow \tau \right) = 1. \quad (6)$$

Otherwise, if there is a strongly consistent estimator g_n , let

$$A_i = \left\{ \omega \in \Omega^*, \lim_{n \rightarrow \infty} g_n(Y_1, \dots, Y_n) \rightarrow \tau_i \right\}, \quad i = 1, 2,$$

for some $\tau_1 \neq \tau_2$. Then, by Equation (6), $P_{\tau_1, \rho}(A_1) = 1$ and $P_{\tau_2, \rho}(A_2) = 1$. Recall that two probability measures P_i on a measurable space (Ω^*, \mathcal{F}) are equivalent if for any $A \in \mathcal{F}$, $P_2(A) = 1$ implies $P_1(A) = 1$ and viceversa. Because $P_{\tau_1, \rho}$ and $P_{\tau_2, \rho}$ are equivalent and $P_{\tau_2, \rho}(A_2) = 1$, we must have $P_{\tau_1, \rho}(A_2) = 1$, which contradicts with $P_{\tau_1, \rho}(A_1) = 1$ because A_1 and A_2 are mutually exclusive. This contradiction implies that strongly consistent estimators do not exist. Consequently, weakly consistent estimators do not exist either because any weakly consistent estimator has a subsequence that is strongly consistent.

Proposition 2.1 implies that τ cannot be estimated well, even when the spatial sample size is extremely large. We note that we did not assume that the spatial domain is a bounded region. Therefore, the proposition holds regardless of the asymptotic framework: fixed-domain, increasing domain or mixed asymptotics.

Finally, we note that if the spatial data are from the model given in Equation (3), then we have no empirical evidence that the data have a non-Gaussian distribution. For example, the histogram of the data $\{Y(s_i), i = 1, \dots, n\}$ is just the histogram of normal data $\{a + \rho Z(s_i), i = 1, \dots, n\}$, where a is the realization of $|Z_0|$ multiplied by τ . Similarly, the sample quantiles of $\{Y(s_i), i = 1, \dots, n\}$ are those of the normal data $\{Z(s_i), i = 1, \dots, n\}$ plus a . Thus, the skewness of the marginal distribution of the process $Y(s)$ is not seen in either the histogram or the normal probability plot. We provide an alternative model in Section 3 that is appropriate for spatial data having skewed marginal distributions.

2.2 ELLIPTICALLY CONTOURED RANDOM FIELDS

We now assume that the underlying spatial process $Y(s)$ is defined by Equation (2). The following proposition implies again an identifiability problem, namely that the distribution of R cannot be identified correctly with probability 1.

PROPOSITION 2.2 For $i = 1, 2$, let P_i be a probability measure under which R has a distribution function F_i with variance 1, and it is independent of the process $Z(s)$ defined by Equation (2). Furthermore, assume that the process $Z(s)$ has the same distribution under both measures P_i , for $i = 1, 2$. If F_1 and F_2 are absolutely continuous with respect to each other, then P_1 and P_2 are equivalent on the paths of $\{Y(s): s \in \mathbb{R}^d\}$.

Next, we consider the special case where $Z(s)$ in Equation (2) is Gaussian, stationary,

with zero-mean and Matérn covariance function given by

$$C(h) = \frac{\sigma^2(\alpha h)^\nu}{2^{\nu-1}\Gamma(\nu)} K_\nu(\alpha h), \quad h \geq 0, \tag{7}$$

where K_ν is the modified Bessel function of the second kind of order $\nu > 0$; see Abramowitz and Stegun (1967, pp. 375–376). The probability distribution of the process $Y(s)$ is determined by the probability distribution of R and the Gaussian distribution of the process $Z(s)$, which depends on the parameters $\theta = (\sigma^2, \alpha, \nu)$. It is known that the parameters σ^2 and α cannot be estimated consistently when $Z(s)$ is observed in a bounded region of \mathbb{R}^d for $d = 1, 2$ and 3 ; see Zhang (2004). The following proposition says that the non-identifiability problem and the inconsistency persist when the process $Y(s)$ defined by Equation (2) is observed.

PROPOSITION 2.3 For $i = 1, 2$, let P_i be a probability measure under which R has a distribution function F_i and $Z(s)$ defined by Equation (2) is stationary, Gaussian, zero-mean and with the Matérn covariance function given in Equation (7) for the parameter $\theta_i = (\sigma_i^2, \alpha_i, \nu)$. If F_1 and F_2 have the same support, $\sigma_1^2 \alpha_1^{2\nu} = \sigma_2^2 \alpha_2^{2\nu}$ and D is a bounded subset of \mathbb{R}^d , for $d = 1, 2, 3$, then the two measures P_1 and P_2 are equivalent on the paths of $\{Y(s) : s \in D\}$.

One immediate corollary of Proposition 2.3 is that F_1 and F_2 cannot be distinguished correctly with probability 1 if $Y(s)$ is observed in a bounded region. Neither can the parameters σ^2 and α be estimated consistently.

3. REMEDIES

3.1 SKEW-GAUSSIAN RANDOM FIELDS

The identifiability problem of the spatial skew-Gaussian random field defined by Equation (1) can be removed by considering the following modification proposed by Zhang and El-Shaarawi (2010):

$$Y(s) = \delta|Z_1(s)| + (1 - \delta^2)^{1/2}Z_2(s), \tag{8}$$

where $Z_1(s)$ and $Z_2(s)$ are zero-mean unit-variance Gaussian random fields, independent of each other, with correlation functions $\rho_{Z_1}(s_1, s_2)$ and $\rho_{Z_2}(s_1, s_2)$, respectively, and $\delta \in (-1, 1)$. The resulting spatial random field $Y(s)$ has mean $E[Y(s)] = \delta(2/\pi)^{1/2}$, variance $\text{Var}[Y(s)] = 1 - (2/\pi)\delta^2$, and covariance function $C_Y(s_1, s_2)$ given by

$$\frac{2}{\pi}\delta^2 \left[(1 - \rho_{Z_1}(s_1, s_2))^2 \right]^{1/2} + \rho_{Z_1}(s_1, s_2) \arcsin(\rho_{Z_1}(s_1, s_2)) - 1 \Big] + (1 - \delta^2)\rho_{Z_2}(s_1, s_2).$$

The marginal distribution of $Y(s)$ is univariate skew-normal according to the definition of Azzalini (1985), that is, $Y(s)$ has probability density function $2\phi(y)\Phi(\alpha y)$, for $y \in \mathbb{R}$, with $\alpha = \delta/(1 - \delta^2)^{1/2} \in \mathbb{R}$. The finite-dimensional joint probability density function of $Y = (Y(s_1), \dots, Y(s_n))^T$ is

$$f_Y(y) = \int_{\mathbb{R}_+^n} \phi_n(y - \delta w; (1 - \delta^2)\Omega_{Z_2}) f_W(w)dw, \tag{9}$$

where the correlation matrices Ω_{Z_1} and Ω_{Z_2} are constructed from $\rho_{Z_1}(s_1, s_2)$ and $\rho_{Z_2}(s_1, s_2)$, respectively. Here $W = |Z_1|$, with $Z_1 \sim N_n(0, \Omega_{Z_1})$, an n -dimensional multivariate normal distribution. However, Equation (9) is difficult to write out explicitly. For example, when $n = 2$, the corresponding joint cumulative distribution function of $W = (W_1, W_2)^\top$ is

$$F_W(w_1, w_2) = \Phi_2(w_1, w_2; \Omega_{Z_1}) - \Phi_2(w_1, -w_2; \Omega_{Z_1}) - \Phi_2(-w_1, w_2; \Omega_{Z_1}) + \Phi_2(-w_1, -w_2; \Omega_{Z_1}),$$

where $\Phi_n(y; \Omega_{Z_1})$ denotes the cumulative distribution function from an n -dimensional multivariate normal distribution, $N_n(0, \Omega_{Z_1})$.

In the particular case where $\Omega_{Z_1} = I_n$, the finite-dimensional joint distribution of $Y = (Y(s_1), \dots, Y(s_n))^\top$ is of a specific unified skew-normal type defined by Arellano-Valle and Azzalini (2006), also called fundamental skew-normal distribution by Arellano-Valle and Genton (2005), with probability density function given by

$$2^n \phi_n(y; \bar{\Omega}) \Phi_n(\delta \bar{\Omega}^{-1} y; I_n - \delta^2 \bar{\Omega}^{-1}), \quad y \in \mathbb{R}^n, \quad (10)$$

and $\bar{\Omega} = \delta^2 I_n + (1 - \delta^2) \Omega_{Z_2}$. If $\Omega_{Z_2} = I_n$, then $\bar{\Omega} = I_n$ and the probability density function given in (10) becomes that of a canonical fundamental skew-normal distribution.

The expression (10) is the likelihood function associated with a realization of size n from the skew-Gaussian random field given in Equation (8) when $Z_1(s)$ is uncorrelated. The cumulative distribution function Φ_n can be evaluated numerically with methods described and implemented by Genz and Bretz (2009). The EM algorithm can also be used to perform inference on the skew-Gaussian random field given in Equation (8) as detailed in Zhang and El-Shaarawi (2010). Notice that the marginal distribution of the model given in Equation (8) has a variance less than one. Hence, to accommodate an unknown variance that may be greater than one, the model given in Equation (8) can be multiplied by a constant, which is equivalent to employing the model

$$Y(s) = \sigma_1 |Z_1(s)| + \sigma_2 Z_2(s),$$

where $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Zhang and El-Shaarawi (2010) applied this model to a spatial data set with skewed marginal distributions.

The unified skew-normal distribution has another stochastic representation based on conditional distributions. Allard and Naveau (2007) used that approach to define spatial skew-normal random fields. Gualtierotti (2005) used a similar idea to define skew-normal processes.

3.2 ELLIPTICALLY CONTOURED RANDOM FIELDS

The identifiability problem of the elliptically contoured random field defined by Equation (2) can be removed by considering the modification

$$Y(s) = (E[(R(s) - E[R(s)])^2])^{-1/2} [R(s) - E[R(s)]] Z(s), \quad (11)$$

where $Z(s)$ is a zero-mean Gaussian random field with covariance function $C_Z(s_1, s_2)$, $R(s)$ is a non-negative random field with finite second moment and correlation $\rho_R(s_1, s_2)$, and $Z(s)$ is independent of $R(s)$. The resulting spatial random field $Y(s)$ has mean $E[Y(s)] = 0$, variance $\text{Var}[Y(s)] = C_Z(s, s)$, and covariance function

$$C_Y(s_1, s_2) = \rho_R(s_1, s_2) C_Z(s_1, s_2).$$

For example, $R(s)$ could be a χ^2 random field; see Ma (2010b). The marginal distribution of $Y(s)$ is elliptically contoured but the finite-dimensional joint distribution of $Y = (Y(s_1), \dots, Y(s_n))^\top$ does not seem to have a known form. However, parameters can be estimated by the EM algorithm, similar to that proposed by Zhang and El-Shaarawi (2010).

4. CONCLUSIONS

We have discussed identifiability issues associated with skew-Gaussian spatial random fields based on the Azzalini and Dalla Valle (1996) multivariate skew-normal distribution and with elliptically contoured random fields. We have also proposed remedies that avoid the unidentifiability. Those ideas can then be combined to define skew-elliptical random fields, that is, by making use of both Equations (8) and (11). A particular case is then skew- t random fields that allow both skewness and heavy tails in the distribution of the spatial data to be modeled; see Azzalini and Genton (2008) and references therein for details about skew- t distributions. Log-normal spatial random fields have also been proposed in the literature and can be extended to log-skew-Gaussian spatial random fields or, more generally, to log-skew-elliptical spatial random fields by means of the log-skew-elliptical multivariate distributions described by Marchenko and Genton (2010).

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APPENDIX: PROOFS

PROOF [Proposition 2.1] Let P_i be the probability measure corresponding to the parameters τ_i, ρ , for $i = 1, 2$. Let $A \in \sigma(Y_i, i = 1, 2, \dots)$, the σ -algebra generated by Y_1, Y_2, \dots . Suppose $P_1(A) = 0$. We show $P_2(A) = 0$. Consequently, P_2 is absolutely continuous with respect to P_1 . Because

$$P_1(A) = E_1[E_1[1_A|Z_0]] = 0,$$

we must have $E_1[1_A|Z_0] = 0$ almost surely with respect to P_1 . Under the measure P_i , given Z_0, Y_1, Y_2, \dots , is a Gaussian sequence with mean $\tau_i|Z_0|$ and covariance function $\text{Cov}(Z_j, Z_k)$. Therefore, for any $z \in \mathbb{R}$

$$E_2[1_A|Z_0 = z] = E_1[1_A|Z_0 = \tau_2 z / \tau_1]. \quad (\text{A.1})$$

If we define $e_i(z) = E_i[1_A|Z_0 = z]$, Equation (A.1) translates into $e_2(z) = e_1(z\tau_2/\tau_1)$. Because $E_1[1_A|Z_0] = 0$ almost surely with respect to P_1 , we have $e_1(z) = E_1[1_A|Z_0 = z] = 0$

and hence $e_2(z) = 0$, for all z . Consequently,

$$P_2(A) = E_2[E_2[1_A|Z_0]] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e_2(z) \exp(-z^2/2) dz = 0.$$

Similarly, we can show that $P_2(A) = 0$ implies $P_1(A) = 0$. This concludes the proof. \blacksquare

PROOF [Proposition 2.2] Let A be a Borel set in \mathbb{R}^d such that $P_1(Y \in A) = 0$, where Y denotes the spatial process $\{Y(s) : s \in \mathbb{R}^d\}$. We show $P_2(Y \in A) = 0$. Write

$$P_1(Y \in A) = \int P_1(Y \in A|R = x) dF_1(x) = \int P_1(Y \in A|R = x) f(x) dF_2(x), \quad (\text{A.2})$$

where $f(x) = dF_1(x)/dF_2(x)$ is the Radon-Nikodym derivative of dF_1 with respect to dF_2 . Therefore, the set

$$\{x : P_1(Y \in A|R = x) f(x) = 0\}$$

has measure 1 with respect to dF_2 . Because dF_2 is absolutely continuous with respect to dF_1 , $f(x) > 0$ almost surely with respect to dF_2 . Consequently,

$$\{x : P_1(Y \in A|R = x) = 0\} = \{x : P_1(Y \in A|R = x) f(x) = 0\},$$

has measure 1 with respect to dF_2 . Because of the independence of R and $Z(s)$, and since the distributions of the process $Z(s)$ under the two measures P_1 and P_2 are identical, we have, for any x ,

$$P_2(Y \in A|R = x) = P_1(Y \in A|R = x).$$

Hence, the set $\{x : P_2(Y \in A|R = x) = 0\}$ has measure 1 with respect to dF_2 . It follows immediately that

$$P_2(Y \in A) = \int P_2(Y \in A|R = x) dF_2(x) = 0.$$

Then, P_2 is absolutely continuous with respect to P_1 . Similarly, we can show that P_1 is absolutely continuous with respect to P_2 . The proof is complete. \blacksquare

PROOF [Proposition 2.3] Let A be a Borel set in \mathbb{R}^d such that $P_1(Y \in A) = 0$, where Y denotes the spatial process $\{Y(s) : s \in D\}$. We show $P_2(Y \in A) = 0$. We have shown in the proof of Proposition 2.2 that the set $\{x : P_1(Y \in A|R = x) = 0\}$ has measure 1 with respect to dF_2 .

Because of the independence of R and $Z(s)$, $P_1(Y \in A|R = x) = P_1(xZ \in A)$. Hence, the set $\{P_1(xZ \in A) = 0\}$ has measure 1 with respect to dF_2 . In addition, since $\sigma_1^2 \alpha_1^{2\nu} = \sigma_2^2 \alpha_2^{2\nu}$, P_1 and P_2 are equivalent on the paths of $Z(s)$, for $s \in D$; see Zhang (2004). It follows that for any x , $P_1(Z \in xA) = 0$ implies $P_2(Z \in xA) = 0$. Thus, the set $\{P_2(xZ \in A) = 0\}$ has measure 1 and therefore

$$P_2(Y \in A) = \int P_2(Y \in A|R = x) dF_2(x) = \int P_2(xZ \in A) dF_2(x) = 0.$$

We have shown that P_2 is absolutely continuous with respect to P_1 . Similarly, we can show that P_1 is absolutely continuous with respect to P_2 . This concludes the proof. \blacksquare

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