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# A note on zonal polynomials

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**Abstract**

In this paper, we correct Lemma 7.2.12 and Theorem 7.2.13 of the book "Aspects of Multivariate Analysis" by R.J. Muirhead. Then, some generalisations of these results are considered.

**Keywords:** Homogeneous polynomials · Random matrices · Zonal polynomials.

**Mathematics Subject Classification:** Primary 62E99 · Secondary 60E10.

## 1. Introduction

Zonal polynomials are of undeniable importance, in both theory and practice. Following the algorithms proposed by Koev and Edelman (2006), zonal polynomials are increasingly being used in various areas of knowledge. Undoubtedly, the initial studies by James (1960, 1961a,b, 1964) and Constantine (1963), among others, laid the foundations in this field. Subsequently, some books such as by Farrell (1985), Takemura (1984) and Forrester (2009) compiled many of these early results and proposed new theoretical considerations and many practical applications.

Recently, it was noticed that the real and complex zonal polynomials (see Farrell, 1985; Takemura, 1984), as well as other polynomials, are particular cases of Jack polynomials; see, e.g., Kaneko (1993), Koev and Edelman (2006) and Forrester (2009). Also, it was observed that most of the properties of the real and complex zonal polynomials are satisfied by Jack polynomials too. In particular, Li and Xue (2009) extended the results proposed by Muirhead (1982) to zonal polynomials of quaternion matrix argument.

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However, the book by Muirhead (1982) marked a watershed in these studies and it has had an undeniable impact on recent generations of mathematicians and statisticians working in the field of multivariate analysis; see Wijsman (1984). Virtually all recent studies that bear upon the question of zonal polynomials have cited Muirhead's book (1982). In particular, Li (1997) calculated the expectation of zonal and invariant polynomials, making use of various results published in Muirhead (1982). Unfortunately, the conclusions drawn by Li (1997) are incorrect. This is because both Lemma 7.2.12 and the proof of Theorem 7.2.13 in Muirhead (1982) are incorrect. Due to the undeniable importance and impact of Muirhead's book, and its influence on current and future studies, the present note proposes corrections to the above-mentioned lemma and theorem.

The paper is organized as follows. In Section 2, we present some useful preliminary aspects. In Section 3, we discuss the main results of this note correcting Lemma 7.2.12 and Theorem 7.2.13 of the book "Aspects of Multivariate Analysis" by R.J. Muirhead, and providing some generalisations of these results. Finally, in Section 3, we provide some conclusions.

## 2. Preliminary Results

In this section, we present some preliminary aspects that are useful for providing the proofs discussed in Section 3.

The Pochhammer symbol is defined as

$$(x)_q = x(x+1)\cdots(x+q-1) = \prod_{i=1}^{q}(x+i-1) = \prod_{i=1}^{q}(x+q-i) = \frac{\Gamma(x+q)}{\Gamma(x)}, \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function. Also, observe that

$$(-x)_q = (-1)^q(x-q+1)_q = \frac{(-1)^q\Gamma(x+1)}{\Gamma(x-q+1)}. \tag{2}$$

Similarly

$$(x)^{(q)} = x(x-1)\cdots(x-q+1) = \prod_{i=1}^{q}(x-i+1) = \prod_{i=1}^{q}(x-q+i). \tag{3}$$

Then, for any function $g\colon \Re \to \Re$,

$$\prod_{i=1}^{q} g(x+i-1) = \prod_{i=1}^{q} g(x+q-i) \tag{4}$$

and

$$\prod_{i=1}^{q} g(x-i+1) = \prod_{i=1}^{q} g(x-q+i). \tag{5}$$

Lemma 2.1 Let $\mathbf{A}$ be a real $m\times m$ positive definite matrix and $\mathrm{Re}(a) > (m-1)/2$. Then, the multivariate gamma function, denoted by $\Gamma_m(a)$, is defined to be

$$\Gamma_m(a) = \int_{\mathbf{A}>0} \mathrm{etr}(-\mathbf{A})(\det \mathbf{A})^{a-\frac{m+1}{2}}(d\mathbf{A}),$$

where $\mathrm{etr}(\cdot) \equiv \exp\mathrm{tr}(\cdot)$, the integral is over the space of positive definite (and hence symmetric) $m\times m$ matrices and $\mathbf{A}>0$ means that $\mathbf{A}$ is a positive definite matrix. Then,

$$\begin{aligned}\Gamma_m(a) &= \pi^{\frac{m(m-1)}{4}}\prod_{i=1}^{m}\Gamma\left(a-\frac{i-1}{2}\right)\\ &= \pi^{\frac{m(m-1)}{4}}\prod_{i=1}^{m}\Gamma\left(a-\frac{m-i}{2}\right).\end{aligned}$$

Proof See Muirhead (1982, Theorem 2.1.12, pp. 62-63), Mathai (1997, Example 1.24, pp. 56-57) and Equation (5). ■

In a similar way to Equation (5), it is readily apparent that for $k_i$ being a non-negative integer and $i = 1,\dots,q$,

$$\prod_{i=1}^{q} g(x \pm k_{q+1-i} - i + 1) = \prod_{i=1}^{q} g(x \pm k_i - q + i). \tag{6}$$

## 3. Zonal Polynomials

In this section, we propose the correct version of Lemma 7.2.12 (p. 256) and the correct proof of Theorem 7.2.13 (pp. 256-258) given in Muirhead (1982).

Lemma 3.1 Let $\mathbf{Z} = \mathrm{diag}\{z_1,\dots,z_m\}$ and $\mathbf{Y} = (y_{ij})$ be an $m\times m$ positive definite matrix. Then,

$$\begin{aligned}C_\kappa\left(\mathbf{Y}^{-1}\mathbf{Z}\right) = d_\kappa z_1^{k_1}\cdots z_m^{k_m} y_{11}^{-(k_m-k_{m-1})} \det\begin{bmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}^{-(k_{m-1}-k_{m-2})} \cdots (\det\mathbf{Y})^{-k_1}\\ + \text{ terms of lower weight in the } z's.\end{aligned}$$

where $\kappa = (k_1,\dots,k_m)$.

Proof If $\mathbf{A}$ is a symmetric matrix with latent roots $a_1,\dots,a_m$, then $\mathbf{A}^{-1}$ is also a symmetric matrix with latent roots $\alpha_1,\dots,\alpha_m$, such that $\alpha_i = 1/a_i$, for $i=1,\dots,m$. Thus, by Constantine (1966, without proof) and Takemura (1984, Lemma 2, p. 54, with proof),

$$(\det\mathbf{A})^n\frac{C_\kappa(\mathbf{A}^{-1})}{C_\kappa(\mathbf{I}_m)} = \frac{C_{\kappa*}(\mathbf{A})}{C_{\kappa*}(\mathbf{I}_m)},$$

where $n$ is any integer $\geq k_1$ and $\kappa^* = (n-k_m, \dots, n-k_1)$. Thus,

$$\begin{aligned} C_\kappa\left(\mathbf{A}^{-1}\right) &= d_\kappa \alpha_1^{k_1} \cdots \alpha_m^{k_m} + \text{ terms of lower weight} \\ &= (\det \mathbf{A})^{-n} s_{\kappa,\kappa^*} C_{\kappa^*}(\mathbf{A}) \\ &= (\det \mathbf{A})^{-n} s_{\kappa,\kappa^*} [d_{\kappa^*} a_1^{n-k_m} \cdots a_m^{n-k_1} + \text{ terms of lower weight}]. \end{aligned}$$

Denoting

$$s_{\kappa,\kappa^*} = \frac{C_\kappa(\mathbf{I}_m)}{C_{\kappa^*}(\mathbf{I}_m)},$$

we have

$$\begin{aligned} C_\kappa\left(\mathbf{A}^{-1}\right) &= (\det \mathbf{A})^{-n} s_{\kappa,\kappa^*} d_{\kappa^*} a_1^{n-k_m-(n-k_{m-1})} (a_1a_2)^{n-k_{m-1}-(n-k_{m-2})} \cdots (a_1a_2\cdots a_m)^{n-k_1} \\ &\quad + \text{terms of lower weight} \\ &= s_{\kappa,\kappa^*} d_{\kappa^*} a_1^{-(k_m-k_{m-1})} (a_1a_2)^{-(k_{m-1}-k_{m-2})} \cdots (a_1a_2\cdots a_m)^{-k_1} \\ &\quad + \text{terms of lower weight} \\ &= s_{\kappa,\kappa^*} d_{\kappa^*} r_1^{-(k_m-k_{m-1})} r_2^{-(k_{m-1}-k_{m-2})} \cdots r_m^{-k_1} \\ &\quad + \text{terms of lower weight}, \end{aligned}$$

where $r_j = \operatorname{tr}_j(\mathbf{A})$. From Equations (39) and (40) given in Muirhead (1982, p. 247), we get

$$\begin{aligned} C_\kappa\left(\mathbf{A}^{-1}\right) &= s_{\kappa,\kappa^*} d_{\kappa^*} \operatorname{tr}_1(\mathbf{A})^{-(k_m-k_{m-1})} \operatorname{tr}_2(\mathbf{A})^{-(k_{m-1}-k_{m-2})} \cdots \operatorname{tr}_m(\mathbf{A})^{-k_1} \\ &\quad + \text{terms of lower weight} \\ &= s_{\kappa,\kappa^*} d_{\kappa^*} a_{11}^{-(k_m-k_{m-1})} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-(k_{m-1}-k_{m-2})} \cdots (\det \mathbf{A})^{-k_1} \\ &\quad + \text{terms of lower weight}. \end{aligned}$$

Now, let $\mathbf{A}^{-1} = \mathbf{Y}^{-1}\mathbf{Z}$. Then, $\mathbf{A} = \mathbf{Z}^{-1}\mathbf{Y}$ and $a_{ij} = z_i^{-1} y_{ij}$. Thus,

$$\begin{aligned} C_\kappa\left(\mathbf{Y}^{-1}\mathbf{Z}\right) &= s_{\kappa,\kappa^*} d_{\kappa^*} (z_1^{-1}y_{11})^{-(k_m-k_{m-1})} \det \begin{bmatrix} z_1^{-1}y_{11} & z_1^{-1}y_{12} \\ z_2^{-1}y_{21} & z_2^{-1}y_{22} \end{bmatrix}^{-k_{m-1}+k_{m-2}} \cdots (\det \mathbf{Z}^{-1}\mathbf{Y})^{-k_1} \\ &\quad + \text{terms of lower weight in the } z's \\ &= s_{\kappa,\kappa^*} d_{\kappa^*} z_1^{k_m} \cdots z_m^{k_1}\ y_{11}^{-(k_m-k_{m-1})} \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-(k_{m-1}-k_{m-2})} \cdots (\det \mathbf{Y})^{-k_1} \\ &\quad + \text{terms of lower weight in the } z's. \end{aligned}$$

Finally, the result is obtained observing that:

i) The Equation (10) given in Constantine (1963) (see also Muirhead, 1982, Equation (i)(1), p. 228) that appears as

$$C_\kappa(\mathbf{Y}) = d_\kappa y_1^{k_1} \cdots y_m^{k_m} + \text{ terms of lower weight}, \tag{7}$$

is a compact form of the expression

$$C_\kappa(\mathbf{Y}) = d_\kappa(y_1^{k_1} \cdots y_m^{k_m} + \cdots + \text{ symmetric terms}) + \text{ terms of lower weight},$$

where "symmetric terms" denote the generic term $y_{i_1}^{k_1} \cdots y_{i_m}^{k_m}$, being $(i_1, \ldots, i_m)$ a permutation of the $m$ integers, namely $1, \ldots, m$, and $\kappa = (k_1, \ldots, k_m)$. Then, for a fixed permutation $(i_1, \ldots, i_m)$, alternately (7) can be written as

$$C_\kappa(\mathbf{Y}) = d_\kappa y_{i_1}^{k_1} \cdots y_{i_m}^{k_m} + \text{ terms of lower weight},$$

or in particular, for our purpose as

$$C_\kappa(\mathbf{Y}) = d_\kappa y_1^{k_m} \cdots, y_m^{k_1} + \text{ terms of lower weight},$$

and

ii) Observing that, the constant associated with "$z_1^{k_1} \cdots z_m^{k_m} + \cdots +$ symmetric terms" of $C_\kappa(\mathbf{Y}^{-1}\mathbf{Z})$, in terms of the latent roots of $\mathbf{YZ}^{-1}$, is $s_{\kappa,\kappa^*} d_{\kappa^*}$, which is denoted by $d_\kappa$.

■

Remark 3.1 Also, observe that the "Hint" in problem 7.5 in Muirhead (1982) is also incorrect.

An application of Lemma 3.1, but in its wrong version, is given by Muirhead (1982, Theorem 7.2.13). Surprisingly, the correct result is obtained. The following results were proposed (without proof) by Constantine (1966) and, simultaneously, with an alternative proof to that given below, by Khatri (1966) and Takemura (1984, Lemma 1, p. 53).

Theorem 3.2 Let $\mathbf{Z}$ be a complex symmetric $m \times m$ matrix with $\text{Re}(\mathbf{Z}) > 0$. Then,

$$\begin{aligned}\int_{\mathbf{X}>0} \text{etr}(-\mathbf{XZ})(\det \mathbf{X})^{a-\frac{m+1}{2}} C_\kappa &= \frac{(-1)^l \Gamma_m(a)}{(-a+(m+1)/2)_\kappa} (\det \mathbf{Z})^{-a} C_\kappa(\mathbf{Z}) \\ &= \frac{\Gamma_m(a)}{(-a+(m+1)/2)_\kappa} (\det \mathbf{Z})^{-a} C_\kappa(-\mathbf{Z}),\end{aligned}$$

for $\text{Re}(a) > k_1 + (m-1)/2$, where $\kappa = (k_1, \ldots, k_m)$ and $l = k_1 + \cdots + k_m$.

Proof First suppose that $\mathbf{Z} > 0$ is real. Let $f(\mathbf{Z})$ denote the integral on the left side of Equation given in Theorem 3.2 and make the change of variable $\mathbf{X} = \mathbf{Z}^{-1/2}\mathbf{Y}\mathbf{Z}^{-1/2}$, with Jacobian $(d\mathbf{X}) = (\det \mathbf{Z})^{-(m+1)/2}(d\mathbf{Y})$. Then we obtain

$$f(\mathbf{Z}) = \int_{\mathbf{Y}>0} \text{etr}(-\mathbf{Y})(\det \mathbf{Y})^{a-\frac{m+1}{2}} C_\kappa \left(\mathbf{Y}^{-1}\mathbf{Z}\right) (d\mathbf{Y})(\det \mathbf{Z})^{-a}. \tag{8}$$

Thus, exactly as in the proof of Theorem 7.2.7 in Muirhead (1982, pp. 256-257), we get

$$f(\mathbf{Z}) = \frac{f(\mathbf{I}_m)}{C_\kappa(\mathbf{I}_m)} C_\kappa(\mathbf{Z})(\det \mathbf{Z})^{-a}.$$

Assuming without loss of generality that $\mathbf{Z} = \operatorname{diag}\{z_1, \dots z_m\}$ and using Definition 7.2.1 (i) given in Muirhead (1982), it follows that

$$f(\mathbf{Z}) = \frac{f(\mathbf{I}_m)}{C_\kappa(\mathbf{I}_m)} (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} + \text{ terms of lower weight.} \tag{9}$$

On the other hand, using the result of Lemma 3.1 in Equation (8), we obtain

$$\begin{aligned} f(\mathbf{Z}) = (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} &\int_{\mathbf{Y}>0} \operatorname{etr}(-\mathbf{Y})(\det \mathbf{Y})^{a-\frac{m+1}{2}} \\ &\times\ y_{11}^{-(k_m-k_{m-1})} \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-(k_{m-1}-k_{m-2})} \cdots \det \mathbf{Y}^{-k_1} (d\mathbf{Y}) \\ &+ \text{terms of lower weight in the } z's. \end{aligned}$$

To evaluate the integral above, set $\mathbf{Y} = \mathbf{T}^\top \mathbf{T}$, where $\mathbf{T}$ is upper-triangular with positive diagonal elements, such that

$$\operatorname{tr}(\mathbf{Y}) = \sum_{i \le j}^m t_{ij}^2, \quad y_{11} = t_{11}^2, \quad \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = t_{11}^2 t_{22}^2, \ \dots, \ \det \mathbf{Y} = \prod_{i=1}^m t_{ii}^2,$$

and, from Theorem 2.1.9 given in Muirhead (1982, p. 60),

$$(d\mathbf{Y}) = 2^m \prod_{i=1}^m t_{ii}^{m+1-i} \bigwedge_{i \le j}^m dt_{ij} = \prod_{i=1}^m \left(t_{ii}^2\right)^{\frac{m-i}{2}} \bigwedge_{i=1}^m dt_{ii}^2 \bigwedge_{i<j}^m dt_{ij}.$$

Hence,

$$\begin{aligned} f(\mathbf{Z}) &= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \prod_{i<j}^m \left( \int_{-\infty}^{\infty} \exp\left(-t_{ij}^2\right) dt_{ij} \right) \\ &\quad \times \prod_{i=1}^m \left( \int_0^\infty \exp\left(-t_{ii}^2\right) \left(t_{ii}^2\right)^{a-k_{m+1-i}-\frac{i-1}{2}-1} dt_{ii}^2 \right) \\ &\quad + \text{terms of lower weight in the } z's \\ &= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m}\, \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^m \Gamma\left( a - k_{m+1-i} - \frac{i-1}{2} \right) \\ &\quad + \text{terms of lower weight in the } z's. \end{aligned}$$

Then, by using Equation (6), we have

$$f(\mathbf{Z}) = (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \Gamma\left(a - k_i - \frac{m-i}{2}\right)$$
$$+ \text{ terms of lower weight in the } z's,$$

which is the result obtained by Khatri (1966, Equation (12)); see also Takemura (1984, Lemma 1, p. 53). Finally, by Equation (2) and las Equation given in Lemma (2.1), we have

$$\begin{aligned}
f(\mathbf{Z}) &= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \frac{(-1)^{k_i}\Gamma\left(a-(m-i)/2\right)}{(-a+(m-i)/2+1)_{k_i}} \\
&\quad + \text{ terms of lower weight in the } z's \\
&= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^l \, \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a-(m-i)/2\right)}{\prod_{i=1}^{m} (-a+(m-i)/2+1)_{k_i}} \\
&\quad + \text{ terms of lower weight in the } z's \\
&= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^l \Gamma_m(a)}{\prod_{i=1}^{m} (-a+(m+1)/2-(i-1)/2)_{k_i}} \\
&\quad + \text{ terms of lower weight in the } z's \\
&= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^l \Gamma_m(a)}{(-a+(m+1)/2)_\kappa} \\
&\quad + \text{ terms of lower weight in the } z's,
\end{aligned}$$

where recall $\kappa = (k_1, \ldots, k_m)$ and $l = k_1 + \cdots + k_m$, and

$$(b)_\kappa = \prod_{i=1}^{m} \left(b - \frac{i-1}{2}\right)_{k_i}.$$

Equating coefficients of $z_1^{k_1} \cdots z_m^{k_m}$ in Equations (9) and (10), it follows that

$$\frac{f(\mathbf{I}_m)}{C_\kappa(\mathbf{I}_m)} = \frac{(-1)^l \Gamma_m(a)}{(-a+(m+1)/2)_\kappa}.$$

Finally, we obtain the desired result for real $\mathbf{Z} > 0$, and it follows for complex $\mathbf{Z}$ with $\mathrm{Re}(\mathbf{Z}) > 0$ by analytic continuation and recalling that $(-1)^l C_\kappa(\mathbf{A}) = C_\kappa(-\mathbf{A})$. ■

Remark 3.2 Observe that Muirhead (1982, penultimate line, p. 257) obtained

$$\begin{aligned}
f(\mathbf{Z}) &= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \Gamma\left(a - k_i - \frac{i-1}{2}\right) \\
&\quad + \text{terms of lower weight in the } z's \\
&= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \frac{(-1)^{k_i}\Gamma\left(a-(i-1)/2\right)}{(-a+(i-1)/2+1)_{k_i}} \\
&\quad + \text{terms of lower weight in the } z's \\
&= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^l \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a-(m-i)/2\right)}{\prod_{i=1}^{m} (-a+(i+1)/2)_{k_i}} \\
&\quad + \text{terms of lower weight in the } z's \\
&= (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^l \Gamma_m(a)}{\prod_{i=1}^{m} (-a+(i+1)/2)_{k_i}} \\
&\quad + \text{terms of lower weight in the } z's \\
&\neq (\det \mathbf{Z})^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^l \Gamma_m(a)}{(-a+(m+1)/2)_\kappa} \\
&\quad + \text{terms of lower weight in the } z's,
\end{aligned}$$

where once again recall $\kappa = (k_1, \dots, k_m)$ and $l = k_1 + \cdots + k_m$.

Corollary 3.3 Let $\mathbf{V}$ be a complex symmetric $m \times m$ matrix with $\operatorname{Re}(\mathbf{V}) > 0$ and $\mathbf{W}$ an arbitrary complex symmetric matrix. Then,

$$\begin{aligned}
\int_{\mathbf{X}>0} \operatorname{etr}(-\mathbf{X}\mathbf{V})(\det \mathbf{X})^{a-\frac{m+1}{2}} C_\kappa\left(\mathbf{W}\mathbf{X}^{-1}\right)(d\mathbf{X}) &= \frac{(-1)^l \Gamma_m(a)}{(-a+(m+1)/2)_\kappa} (\det \mathbf{V})^{-a} C_\kappa(\mathbf{V}\mathbf{W}) \\
&= \frac{\Gamma_m(a)}{(-a+(m+1)/2)_\kappa} (\det \mathbf{V})^{-a} C_\kappa(-\mathbf{V}\mathbf{W}),
\end{aligned}$$

for $\operatorname{Re}(a) > k_1 + (m-1)/2$, where again $\kappa = (k_1, \dots, k_m)$ and $l = k_1 + \cdots + k_m$.

Proof Observe that if $\mathbf{V} = \mathbf{I}_m$ in Equation given in Corollary 3.3, we obtain Equation (8). For the general case substitute $\mathbf{V}^{1/2}\mathbf{X}\mathbf{V}^{1/2}$ for $\mathbf{X}$ in Equation (8) with the Jacobian of the transformation $|\mathbf{V}|^{(m+1)/2}$. ■

Remark 3.3 One reviewer of this paper suggested that Lemma 3.1, Theorem 3.2 and Corollary 3.3 can be generalised to Jack polynomials. In fact, for example, in Corollary 2.4 and Proposition 2.3 of Ratnarajah et al. (2005b) were proposed (without proof) the versions of Theorem 3.2 and Corollary 3.3, respectively, in the complex case. Similarly, the generalisation of Lemma 3.1 and Corollary 3.3 for the quaternion case were proposed (without proof) in Lemma 3.3 and Theorem 3.2 of Li and Xue (2009), respectively. So, these and many other properties of Jack polynomials, the associated general hypergeometric functions with one and two matrix argument and invariant polynomials with two matrix argument for normed division algebras are studied in detail by Díaz-García (2009).

## 4. Conclusions

Let us stress that the aim of the present note is not to disparage the importance of Muirheads book, but rather to correct the certain deficiencies we believe to have identified. Thus, we help prevent, or minimize, erroneous conclusions being drawn on the basis of this note, in both current and future work.

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