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Research Paper

Hypotheses testing for comparing means and variances of correlated responses in the symmetric non-normal case

Manuel Galea[1,*] and Filidor Vilca[2]

[1]Departamento de Estadística, Pontificia Universidad Católica de Chile, Santiago, Chile
[2]Departamento de Estatística, Universidade Estadual de Campinas, Campinas, Brazil

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Abstract

In this paper, we consider hypothesis testing for the equality of means and variances of correlated responses with non-normal distributions. Specifically, we assume that the responses follow a symmetric multivariate distribution. Wald type statistics are considered which are asymptotically distributed according to a chi-square distribution. Statistics are based on the sample mean and the sample covariance matrix. Applications are made for comparing measurement methods and the performance of investment portfolios.

Keywords: Hypotheses testing · Measuring devices · Performance of portfolios · Symmetric distributions.

Mathematics Subject Classification: Primary 62F03 · Secondary 60E05.

1. Introduction

The problem of testing the equality of means and/or variances appears frequently in the analysis of experimental data. Indeed, suppose that a $p \times 1$ random vector, Y, has mean $\boldsymbol{\mu}$, a $p \times 1$ vector, with elements μ_i, and covariance matrix $\boldsymbol{\Sigma}$, a $p \times p$ matrix, with elements σ_{ij}. Sometimes we are interested in simultaneously testing the hypotheses

$$\mathrm{H}_{01}\colon \mu_1 = \mu_2 = \cdots = \mu_p \quad \text{and} \quad \mathrm{H}_{02}\colon \sigma_{11} = \sigma_{22} = \cdots = \sigma_{pp}.$$

Depending on the application, these hypotheses can be tested jointly (H_{03}: $\mathrm{H}_{01} \cap \mathrm{H}_{02}$, say) or separately. Moreover, given that one rejects one of these hypothesis, it is frequent to consider sub-hypotheses for testing equality of a subset of means and/or variances.

The problem of testing the hypothesis H_{02}, for example, has been discussed by several authors. Pitman (1939) and Morgan (1939) proposed a test for the bivariate case. Assuming that the responses are equicorrelated, Wilks (1946) developed the likelihood ratio test and Han (1968) proposed four tests for H_{02}. Choi and Wette (1972) suggested a test based on the multiple correlation coefficient. Harris (1985) developed Wald-type tests, based on the sample covariance matrix. More recently, Reza (1993) considered the likelihood ratio test for the same hypothesis. In all these articles, it is assumed that the sample comes from a multivariate normal distribution.

*Corresponding author. M. Galea. Departamento de Estadística, Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Macul, Santiago, Chile. Email: mgalea@mat.puc.cl

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Harris (1985), Cohen (1986) and Wilcox (1989) proposed several robust tests for H_{02}. In particular, Cohen (1986) proposed two procedures based on a nonparametric bootstrap.

Although the normality assumption is adequate in many situations, it is not appropriate when the data comes from a distribution with heavier tails than the normal. This suggests considering a wider class of distributions. For example, Lange et al. (1989) recommended the t-distribution, while Little (1988) considered contaminated normal distributions. Both models incorporate additional parameters, which allow the kurtosis of the distribution to be fitted. These distributions are elements of a broader class of parametric models (that preserve the symmetric structure) known as elliptic distributions and widely investigated in the statistical literature; see, e.g., Fang et al. (1990) and Fang and Zhang (1990).

The object of this paper is to consider hypothesis testing assuming that we have a random sample from a multivariate symmetric distribution with finite fourth moments. For testing the hypothesis H_{01}, H_{02} and H_{03} we use Wald's statistic. Also, we extend the tests considered in Choi and Wette (1972).

The paper is organized as follows. In Section 2, we provide some aspects of multivariate symmetric distributions. In Section 3, we obtain Wald-type statistics for testing linear hypotheses about the means and/or variances under symmetric distributions. In Section 4, we present applications to the comparison of measuring devices and to the comparison of Sharpe measures. Also, in this section, we consider moment estimators for the Sharpe ratio and obtain an expression for the asymptotic covariance matrix, generalizing the results of Jobson and Korkie (1981). Finally, in Section 5, we sketch some conclusions.

2. Symmetric Multivariate Distributions

We say that the $p \times 1$ dimensional random vector Y has a symmetric distribution with location parameter $\boldsymbol{\mu}$ a $p \times 1$ vector and a $p \times p$ scale matrix $\boldsymbol{\Lambda}$, if its density is given by:

$$f(y; \boldsymbol{\mu}, \boldsymbol{\Lambda}) = |\boldsymbol{\Lambda}|^{-1/2} g[(y-\boldsymbol{\mu})^\top \boldsymbol{\Lambda}^{-1}(y-\boldsymbol{\mu})], \quad y \in \mathbb{R}^p, \tag{1}$$

where the function $g\colon \mathbb{R} \to [0,\infty)$ is such that $\int_0^\infty u^{p-1} g(u^2)du < \infty$. The function g is known as the density generator. For a vector Y distributed according to the density given in Equation (1), we use the notation $Y \sim \mathrm{S}_p(\boldsymbol{\mu}, \boldsymbol{\Lambda}; g)$ or simply $\mathrm{S}_p(\boldsymbol{\mu}, \boldsymbol{\Lambda})$. Kelker (1970) and Cambanis et al. (1981) discussed many properties of the symmetric (or elliptic) distributions. The characteristic function is given by

$$\phi(\boldsymbol{t}) = \exp(i\boldsymbol{t}^\top \boldsymbol{\mu}) \varphi(\boldsymbol{t}^\top \boldsymbol{\Lambda} \boldsymbol{t})$$

for some function φ, where $i = \sqrt{-1}$. Provided they exists, $\mathrm{E}(Y) = \boldsymbol{\mu}$ and $\mathrm{Var}(Y) = c_g \boldsymbol{\Lambda} = \boldsymbol{\Sigma}$, where $c_g = -2\varphi^{(1)}(0)$ is a positive constant. The random vector Y has the representation

$$Y \overset{\mathrm{d}}{=} \boldsymbol{\mu} + R\boldsymbol{A}U,$$

where $\overset{\mathrm{d}}{=}$ means equal in distribution, R is a positive random variable, U has the uniform distribution on $\boldsymbol{u}^\top \boldsymbol{u} = 1$, R and U are independent, and $\boldsymbol{A}$ is a nonsingular matrix such that $\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{\Lambda}$. The moments of R are related to the characteristic function. For example,

(i) $\mathrm{E}(R^2) = -2p\varphi^{(1)}(0)$,
(ii) $\mathrm{E}(R^4) = 4p(p+2)\varphi^{(2)}(0)$,
(iii) $\mathrm{E}(R^6) = -8p(p+2)(p+4)\varphi^{(3)}(0)$ and
(iv) $\mathrm{E}(R^8) = 16p(p+2)(p+4)(p+6)\varphi^{(4)}(0)$.

If Y has finite fourth moments each component of Y has zero skewness and the same kurtosis, given by

$$3\left\{\frac{\varphi^{(2)}(0)}{[\varphi^{(1)}(0)]^2}-1\right\}=3\kappa,$$

where 3κ is called the kurtosis parameter of $\mathrm{S}_p(\boldsymbol{\mu},\boldsymbol{\Lambda})$.

In the case where $\boldsymbol{\mu}=\mathbf{0}$ and $\boldsymbol{\Lambda}=\boldsymbol{I}_p$ (identity matrix of dimension p), we obtain the spherical family of densities.

Let $Y_i=(y_{i1},\ldots,y_{ip})^\top$, for $i=1,\ldots,N$, be the i-th $p\times 1$ response vector of a random sample from a p-variate symmetric distribution $\mathrm{S}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_p)^\top$ and $\boldsymbol{\Sigma}=(\sigma_{ij})$, for $i,j=1,\ldots,p$. We are interested in simultaneously testing the hypotheses

$$\mathrm{H}_{01}\colon \mu_1=\mu_2=\cdots=\mu_p \quad \text{and} \quad \mathrm{H}_{02}\colon \sigma_{11}=\sigma_{22}=\cdots=\sigma_{pp}.$$

Let $\boldsymbol{\sigma}=(\sigma_{11},\ldots,\sigma_{pp})^\top$ and $\boldsymbol{\theta}=(\boldsymbol{\mu}^\top,\boldsymbol{\sigma}^\top)^\top$. More generally, we can test the linear hypothesis H: $\boldsymbol{A}\boldsymbol{\theta}=\boldsymbol{a}$.

3. Hypothesis Testing

In this section, we develop some asymptotic tests for the hypotheses H_{01}, H_{02} and H_{03}, assuming that we have a random sample of size N from a multivariate symmetric distribution. First, we test the hypotheses using Wald's statistic. Afterwards, we present an alternative method for testing the hypotheses H_{01}, H_{02} and H_{03} using certain transformations of the data. Under normality, Choi and Wette (1972) tested the hypothesis H_{02}. Note that H_{02} can be written as $\mathrm{H}_{02}\colon \boldsymbol{A}_1\boldsymbol{\sigma}=\mathbf{0}$, where

$$\boldsymbol{A}_1=\begin{pmatrix} 1 & 0 & 0 & \ldots & -1\\ 0 & 1 & 0 & \ldots & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & -1\end{pmatrix}$$

is a $(p-1)\times p$ contrast matrix. Of course, we can also write $\mathrm{H}_{01}\colon \boldsymbol{A}_1\boldsymbol{\mu}=\mathbf{0}$.

3.1 Wald-type tests

Let

$$\boldsymbol{S}=(s_{ij})=\frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})(Y_i-\bar{Y})^\top$$

denotes the $p\times p$ sample covariance matrix and $V=(s_{11},\ldots,s_{pp})^\top$ denotes a $p\times 1$ vector of diagonal elements of the matrix $\boldsymbol{S}$. We consider a moment estimator of $\boldsymbol{\theta}$, say $\tilde{\boldsymbol{\theta}}=(\bar{Y}^\top,V^\top)^\top$, where $\bar{Y}=(1/N)\sum_{i=1}^{N}Y_i$. From Anderson (1993), we know that

$$\sqrt{n}\,(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta})\xrightarrow{\mathrm{d}}\mathrm{N}_{2p}(\mathbf{0},\boldsymbol{\Omega})\ ,$$

where $\xrightarrow{\text{d}}$ means convergence in distribution and

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_{\mu\mu} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{\sigma\sigma} \end{pmatrix},$$

with $n = N - 1$, $\boldsymbol{\Omega}_{\mu\mu} = \boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}_{\sigma\sigma} = 2(1+\kappa)\boldsymbol{\Sigma} \oplus \boldsymbol{\Sigma} + \kappa\boldsymbol{\sigma}\boldsymbol{\sigma}^\top$, where $\oplus$ denotes the Hadamard product. If we estimate $\boldsymbol{\Sigma}$ by $\boldsymbol{S}$, Wald's statistic for testing H_{03} is given by

$$W_{03} = n\bar{Y}^\top \boldsymbol{A}_1^\top (\boldsymbol{A}_1 \boldsymbol{S} \boldsymbol{A}_1^\top)^{-1} \boldsymbol{A}_1 \bar{Y} + nV^\top \boldsymbol{A}_1^\top (\boldsymbol{A}_1 \hat{\boldsymbol{\Omega}}_{\sigma\sigma} \boldsymbol{A}_1^\top)^{-1} \boldsymbol{A}_1 V = W_{01} + W_{02},$$

where $\hat{\boldsymbol{\Omega}}_{\sigma\sigma} = 2(1+\kappa)\boldsymbol{S} \oplus \boldsymbol{S} + \kappa VV^\top$. Thus we reject at level α if $W_{03} > \chi^2_{1-\alpha}(2(p-1))$, where $\chi^2_{1-\alpha}(2(p-1))$ denotes the $100(1-\alpha)\%$ percentile of a chi-square distribution with $2(p-1)$ degrees of freedom. For testing H_{01}, Wald's statistic is W_{01} which converges in distribution, under H_{01}, to a $\chi^2(p-1)$. Then we reject at level α if $W_{01} > \chi^2_{1-\alpha}(p-1)$. Finally, for testing H_{02}, Wald's statistic is W_{02} and we reject at level α if $W_{02} > \chi^2_{1-\alpha}(p-1)$. Note that, in the normal case, $\kappa = 0$ and

$$W_{02} = \frac{n}{2} V^\top \boldsymbol{A}_1^\top (\boldsymbol{A}_1 (\boldsymbol{S} \oplus \boldsymbol{S}) \boldsymbol{A}_1^\top)^{-1} \boldsymbol{A}_1 V,$$

which coincides with the W statistic given in Equation (2.3) of Harris (1985).

3.2 An alternative test

As in Harris (1985), let $L = (l_1, \ldots, l_p)^\top$, with $l_j = \log s_{jj}$, for $j = 1, \ldots, p$ and $\tilde{\boldsymbol{\gamma}} = h(\bar{Y}, V) = (\bar{Y}^\top, L^\top)^\top$. Then, using the delta method, we have that

$$\sqrt{n}\,(\tilde{\gamma} - \gamma) \xrightarrow{\text{d}} \mathrm{N}_{2p}(\mathbf{0}, \boldsymbol{\Psi}),$$

where

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_2 \end{pmatrix},$$

with $\boldsymbol{\Psi}_2 = 2(1+\kappa)\boldsymbol{\rho} \oplus \boldsymbol{\rho} + \kappa\mathbf{1}\mathbf{1}^\top$, $\boldsymbol{\rho}$ being the correlation matrix in the population, and $\mathbf{1}$ being a $p \times 1$ vector of ones. Then, Wald's statistic for testing H_{03} is given by

$$W_{L03} = W_{01} + nL^\top \boldsymbol{A}_1^\top (\boldsymbol{A}_1 \hat{\boldsymbol{\Psi}}_2 \boldsymbol{A}_1^\top)^{-1} \boldsymbol{A}_1 L = W_{01} + W_{L02},$$

with $\hat{\boldsymbol{\Psi}}_2 = 2(1+\kappa)\boldsymbol{R} \oplus \boldsymbol{R} + \kappa\mathbf{1}\mathbf{1}^\top$, where $\boldsymbol{R}$ is the sample correlation matrix. Under H_{03}, and for large samples, W_{L03} has a χ^2 distribution with $2(p-1)$ degrees of freedom. Note that, in the normal case, $\kappa = 0$ and W_{L02} coincides with W_L given in Harris (1985).

3.3 Estimation of the kurtosis parameter

To implement the previous tests it is necessary to know or estimate the kurtosis parameter κ. First, we note that

$$\mathrm{E}(\{(Y - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (Y - \boldsymbol{\mu})\}^2) = p(p+2)(\kappa+1).$$

Let

$$b_{2,p} = \frac{1}{N}\sum_{i=1}^{N}\{(Y_i - \bar{Y})^\top \boldsymbol{S}^{-1}(Y_i - \bar{Y})\}^2.$$

Then, we see that $b_{2,p}$ converges in probability to $p(p+2)(\kappa+1)$, from which it follows that $b_{2,p}/[p(p+2)] - 1$ converges in probability to κ. Thus, a consistent estimator of κ (Mardia, 1970) is given by

$$\tilde{\kappa} = \frac{b_{2,p}}{p(p+2)} - 1.$$

The convergence of the tests discussed above is valid when κ is replaced by the consistent estimator $\tilde{\kappa}$. From Henze (1994) and Seo and Toyama (1996), the limit distribution of $\sqrt{N}\,(\tilde{\kappa}-\kappa)$ is a normal distribution with variance given by

$$\sigma_\kappa^2 = \frac{1}{p(p+2)}(p+4)(p+6)(\kappa^{(4)}+1) - \frac{4}{p}(p+4)(\kappa^{(3)}+1)(\kappa+1) + \frac{4}{p}(p+2)(\kappa+1)^3 - (\kappa+1)^2, \quad (2)$$

where $\kappa^{(j)} = (\varphi^{(j)}(0)/\{\varphi^{(1)}(0)\}^j) - 1$, for $j = 3, 4$.

Let

$$b_{j,p} = \frac{1}{N}\sum_{i=1}^{N}\{(Y_i - \bar{Y})^\top \boldsymbol{S}^{-1}(Y_i - \bar{Y})\}^j,$$

for $j = 3, 4$. Then we have that (Seo and Toyama, 1996) $(b_{j,p}/c_j) - 1$ converges in probability to $\kappa^{(j)}$, where $c_3 = p(p+2)(p+4)$ and $c_4 = c_3(p+6)$. Then, a consistent estimator of $\kappa^{(j)}$ is given by $\tilde{\kappa}^{(j)} = (b_{j,p}/c_j) - 1$, for $j = 3, 4$; see Seo and Toyama (1996) for an estimator with bias correction, and Maruyama and Seo (2003) for some generalizations of Seo and Toyama's results.

A confidence interval for κ with confidence coefficient $1-\alpha$ is approximately

$$\left(\tilde{\kappa} - z_{\alpha/2}\sqrt{\frac{\tilde{\sigma}_\kappa^2}{N}},\ \ \tilde{\kappa} + z_{\alpha/2}\sqrt{\frac{\tilde{\sigma}_\kappa^2}{N}}\right),$$

where $\tilde{\sigma}_\kappa^2$ is obtained substituting $\tilde{\kappa}$ by κ and $\tilde{\kappa}^{(j)}$ by $\kappa^{(j)}$, for $j = 3, 4$, in Equation (2), and $z_{\alpha/2}$ is the $(1-\alpha/2)100\%$ percentile of the standard normal distribution.

3.4 Generalized Choi-Wette test

In this section we propose an alternative test for H_{03} (and consequently for H_{01} and H_{02}), now assuming that the covariances between responses are all equal. That is, we assume that $\sigma_{ij} = \sigma^2$, for $i \neq j$. We extend the test proposed by Choi and Wette (1972) for H_{02} under normality. We generalize this tests to the class of symmetric distributions with finite fourth moments.

Under normality, Choi and Wette (1972) proposed a test for H_{02}, extending Pitman's test; see Pitman (1939). Following Choi and Wette (1972), let

$$x_{i1} = \sum_{j=1}^{p} y_{ij} \quad \text{and} \quad x_{ij} = p y_{ij} - x_{i1}, \;\; j = 2, \ldots, p, \;\; i = 1, \ldots, N. \tag{3}$$

Then, note that

$$\mathrm{Cov}(x_{i1}, X_{i2}) = \boldsymbol{V}_{12} = \left(p\sigma_{22} - \sum_{k=1}^{p} \sigma_{kk}, \ldots, p\sigma_{pp} - \sum_{k=1}^{p} \sigma_{kk} \right),$$

where $X_{i2} = (x_{i2}, \ldots, x_{ip})^\top$, for $i = 1, \ldots, N$, from which it follows that (Choi and Wette, 1972; Han, 1968) H_{02} is equivalent to $\boldsymbol{V}_{12} = \mathbf{0}$.

Let $X_i = (x_{i1}, \ldots, x_{ip})^\top$. Then, the transformation defined in Equation (3) can be written in matrix notation as $X_i = \boldsymbol{T} Y_i$, where $\boldsymbol{T} = \mathbf{1}_{(-)} \mathbf{1}^\top + p\,\boldsymbol{D}$, with $\boldsymbol{D} = \mathrm{diag}\{0, 1, \ldots, 1\}$ a diagonal matrix, $p \times p$, $\mathbf{1}_{(-)} = (1, -1, \ldots, -1)^\top$ and $\mathbf{1} = (1, \ldots, 1)^\top$, both $p \times 1$ vectors.

From Equation (3) and by using properties of elliptic distributions, we have that $X_i \sim \mathrm{S}_p(\boldsymbol{\mu}_x, \boldsymbol{\Lambda}_x)$, where $\boldsymbol{\mu}_x = p(\bar{\mu}, \mu_2 - \bar{\mu}, \mu_3 - \bar{\mu}, \ldots, \mu_p - \bar{\mu})^\top$ and

$$\boldsymbol{\Sigma}_x = \begin{pmatrix} v_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix} = \boldsymbol{V},$$

with $\bar{\mu} = (1/p) \sum_{j=1}^{p} \mu_j$, v_{11} being the variance of x_1, and $\boldsymbol{V}_{22}$ being the covariance matrix of $X_2 = (x_2, x_3, \ldots, x_p)^\top$. Let $\bar{R}$ be the multiple correlation coefficient between x_1 and the variables $x_2, \ldots, x_p$. We know that (Muirhead, 1982)

$$\bar{R} = \left\{ \frac{\boldsymbol{V}_{12} \boldsymbol{V}_{22}^{-1} \boldsymbol{V}_{21}}{v_{11}} \right\}^{1/2}.$$

Thus, testing the hypothesis H_{02} is equivalent to testing H: $\bar{R} = 0$. Note that the equivalence only holds when all the σ_{ij}, for $i \neq j$, are equal. We consider now the hypothesis H_{03}, which is equivalent to H_{03}: $\boldsymbol{\mu}_{x(2)} = \mathbf{0}$ and $\bar{R} = 0$, where $\boldsymbol{\mu}_{x(2)} = p(\mu_2 - \bar{\mu}, \mu_3 - \bar{\mu}, \ldots, \mu_p - \bar{\mu})^\top$.

Let

$$\boldsymbol{S}_x = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X})(X_i - \bar{X})^\top,$$

the covariance matrix of the random sample $X_1, \ldots, X_N$ from a symmetric distributions $\mathrm{S}_p(\boldsymbol{\mu}_x, \boldsymbol{\Lambda}_x)$. Consider the following partition of $\boldsymbol{S}_x$,

$$\boldsymbol{S}_x = \begin{pmatrix} s_{x11} & \boldsymbol{S}_{x12} \\ \boldsymbol{S}_{x21} & \boldsymbol{S}_{x22} \end{pmatrix}.$$

Then, from Muirhead (1982), we have that, if $\bar{R} = 0$,

$$\sqrt{n} \begin{pmatrix} \bar{X} - \boldsymbol{\mu}_x \\ \boldsymbol{S}_{x12} \end{pmatrix} \xrightarrow{\mathrm{d}} \mathrm{N}_{2(p-1)}(\mathbf{0}, \boldsymbol{\Delta}),$$

where

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{\Sigma}_x & \boldsymbol{0}^\top \\ \boldsymbol{0} & (1+\kappa)\boldsymbol{I}_{p-1} \end{pmatrix}.$$

Thus, under H_{03},

$$n\bar{X}_2^\top \boldsymbol{V}_{22}^{-1}\bar{X}_2 + \frac{n}{1+\kappa}\boldsymbol{S}_{x12}^\top \boldsymbol{S}_{x12} \xrightarrow{\text{d}} \chi^2(2(p-1)),$$

where $\bar{X}_2 = (\bar{x}_2, \ldots, \bar{x}_p)^\top$. On the other hand,

$$nR_n^2 = n\boldsymbol{S}_{x12}^\top \boldsymbol{S}_{x12} + O_p(n^{-1/2}),$$

(Muirhead, 1982), where

$$R_n = \left\{ \frac{\boldsymbol{S}_{x12}\boldsymbol{S}_{x22}^{-1}\boldsymbol{S}_{x21}}{s_{x11}} \right\}^{1/2}$$

is the sample multiple correlation coefficient. By using $\boldsymbol{S}_{x22}$ as a consistent estimator of $\boldsymbol{V}_{22}$ and applying Slusky's theorem, we have that the statistic

$$W_{03}^* = n\bar{X}_2^\top \boldsymbol{S}_{x22}^{-1}\bar{X}_2 + \frac{n}{(1+\kappa)}\, R_n^2 = W_{01}^* + W_{02}^*$$

converges in distribution to $\chi^2(2(p-1))$ under H_{03}. Thus, we reject H_{03} at level α if $W_{03}^* > \chi^2_{1-\alpha}(2(p-1))$. We also have an approximate test for testing H_{02} (or H) when we reject at level α if $W_{02}^* > \chi^2_{1-\alpha}(p-1)$.

Under normality we have an exact test (Muirhead, 1982) for testing H: $\bar{R} = 0$. In fact, under H, we have

$$F = \frac{(N-p)}{(p-1)}\frac{R_n^2}{(1-R_n^2)} \sim F(p-1, N-p).$$

4. Applications

In this section we present two applications of the test discussed in the previous section. The first is related to the problem of comparing measuring devices, while the second compares Sharpe measures.

4.1 Comparing measuring devices

Comparing measuring devices which vary in pricing, speed, and other features such as efficiency, has been of growing interest in many engineering and scientific applications. Grubbs (1948, 1973, 1983) proposed a model for N items, each measured by p instruments, such that $y_{ij} = \alpha_j + x_i + \varepsilon_{ij}$, where y_{ij} represents the measurement of the i-th item with the j-th instrument, x_i is the characteristic of interest in the i-th experimental unit and α_j is called additive bias, for $j = 1, \ldots, p$ and $i = 1, \ldots, N$. In the literature it is assumed that x_i and ε_{ij} are independent with normal distributions $\text{N}(\mu_x, \phi_x)$ and $\text{N}(0, \phi_j)$, respectively. In order to allow this model to be identifiable we may consider $\alpha_1 = 0$; see, e.g., Shyr and Gleser (1986), Bedrick (2001) and Christensen and Blackwood (1993). However, in this work, we assume the transformation $z_i = x_i - \mu_x$, for $i = 1, \ldots, N$ (Theobald and Mallison, 1978), so that Grubbs' model is expressed in an alternative form as

$$Y_i = \boldsymbol{\mu} + \mathbf{1}_p z_i + \boldsymbol{\varepsilon}_i, \qquad i = 1, \ldots, N,$$

where $Y_i = (y_{i1}, \ldots, y_{ip})^\top$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^\top$, with $\mu_j = \alpha_j + \mu_x$ and $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \ldots, \varepsilon_{ip})^\top$.

The symmetric model is obtained by considering the random vectors $Y_1, \ldots, Y_N$ as independent and identically distributed $\mathrm{S}_p(\boldsymbol{\mu}, \boldsymbol{\Lambda})$, where $\boldsymbol{\mu} = \boldsymbol{\alpha} + \mathbf{1}_p\mu_x$ and covariance matrix $\boldsymbol{\Sigma} = c_g\boldsymbol{\Lambda} = \phi_x \mathbf{1}_p\mathbf{1}_p^\top + D(\boldsymbol{\phi})$, with $D(\boldsymbol{\phi}) = \mathrm{diag}\{\phi_1, \ldots, \phi_p\}$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top$.

In the context of Grubb's model, the quality of the measurements is assessed in terms of the additive bias and the precision (inverse of the variance) of the different instruments. Thus, one hypothesis of interest to evaluate the exactness of the measurements made by different instruments is H_{01}: $\mu_1 = \mu_2 = \cdots = \mu_p$. To compare the precision of the instruments, the hypothesis is H_{02}: $\phi_1 = \cdots = \phi_p$. The hypothesis that considers both conditions is H_{03}: $\mathrm{H}_{01} \cap \mathrm{H}_{02}$. Note that, in this case, $\sigma_{ij} = \phi_x$ for all $i \neq j$, so for testing the hypothesis of interest we can use the results of Subsection 3.4.

Alternatively, in the presence of a gold standard (that is to say, an instrument for measuring the characteristic of interest without error) St. Laurent (1998), proposed the model $y_{ij} = x_i + \varepsilon_{ij}$, to assess the degree of agreement between the measurements made by p approximate methods and the gold standard, for $i = 1, \ldots, N$ and $j = 1, \ldots, p$. In this case, x_i corresponds to the measurement of the gold standard in the i-th experimental unit. The model is expressed in an alternative form as

$$Y_i = \mathbf{1}_p x_i + \boldsymbol{\varepsilon}_i, \quad i = 1, \ldots, N,$$

where, $\mathrm{E}(x_i) = \mu$, $\mathrm{Var}(x_i) = \phi_x$, $\mathrm{E}(\boldsymbol{\varepsilon}_i) = \mathbf{0}$, $\mathrm{Var}(\boldsymbol{\varepsilon}_i) = \boldsymbol{\Sigma}$, and x_i is independent of $\boldsymbol{\varepsilon}_i$. Note that in this model it is possible that the random measurement errors of the approximate methods, ε_{ij}, are correlated.

St. Laurent (1998) used $\rho_j = \phi_x/(\phi_x + \sigma_{jj})$ to assess the degree of agreement between the measurements made with the j-th approximate method and the gold standard. In this case the hypothesis of interest is H: $\rho_1 = \rho_2 = \cdots = \rho_p$. In order to do this, St. Laurent (1998) defines the differences, $D_i = Y_i - \mathbf{1}_p x_i$, for $i = 1, \ldots, N$. Note that $D_1, \ldots, D_N$ are random i.i.d. vectors with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$. Under the normality assumption we have that (St. Laurent, 1998) the maximum likelihood estimators of ρ_j and $\boldsymbol{\Sigma}$ are given by

$$\hat{\rho}_j = \frac{1}{1 + \hat{\sigma}_{jj}/\hat{\phi}_x}, \; j = 1, \ldots, p, \;\; \text{and} \;\; \hat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{i=1}^{N} D_i D_i^\top,$$

where

$$\hat{\phi}_x = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 \quad \text{and} \quad \frac{1}{N}\hat{\sigma}_{jj} = \frac{1}{N}\sum_{i=1}^{N}(y_{ij} - x_i)^2, \quad j = 1, \ldots, p.$$

Finally, since $\rho_1 = \rho_2 = \cdots = \rho_p$ if and only if $\sigma_{11} = \sigma_{22} = \cdots = \sigma_{pp}$, testing H is equivalent to testing H_{02}: $\boldsymbol{A}_1\boldsymbol{\sigma} = \mathbf{0}$, so we can use the test statistics derived in Subsections 3.1 and 3.2 for testing H: $\rho_1 = \rho_2 = \cdots = \rho_p$, under symmetric non-normal measurements, i.e., $\mathrm{S}_p(\mathbf{0}, \boldsymbol{\Sigma})$ distributions.

4.2 Comparing Sharpe measures

As in Jobson and Korkie (1981), consider the general situation in which the relative performance of a finite number of portfolios is to be evaluated. Let y_{ij} represents the return premium from the j-th portfolio in period i, for $j = 1, \ldots, p$ and $i = 1, \ldots, N$. In this case, $Y_i = (y_{i1}, \ldots, y_{ip})^\top$ corresponds to a $p \times 1$ random vector of excess returns of p portfolios in the i-th period. The performance measure of Sharpe, for portfolio j, is defined by

$$\tau_j = \frac{\mu_j}{\sqrt{\sigma_{jj}}}, \; j = 1, \ldots, p.$$

The moment estimator of τ_j is $\tilde{\tau}_j = \bar{y}_j/\sqrt{s_{jj}}$, for $j = 1, \ldots, p$. Then, using the delta method, we have

$$\sqrt{n}\,(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}) \xrightarrow{\text{d}} \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Psi}) \ ,$$

where $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_p)^\top$, $\tilde{\boldsymbol{\tau}} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_p)^\top$ and

$$\boldsymbol{\Psi} = \boldsymbol{D}_1 \boldsymbol{\Sigma} \boldsymbol{D}_1^\top + \boldsymbol{D}_2\{2(1+\kappa)\boldsymbol{\Sigma} \oplus \boldsymbol{\Sigma} + \kappa \boldsymbol{\sigma}\boldsymbol{\sigma}^\top\}\boldsymbol{D}_2^\top,$$

with $\boldsymbol{D}_1 = \mathrm{diag}\{1/\sqrt{\sigma_{11}}, \ldots, 1/\sqrt{\sigma_{pp}}\}$ and $\boldsymbol{D}_2 = -(1/2)\ \mathrm{diag}\{\tau_1/\sqrt{\sigma_{11}}, \ldots, \tau_p/\sqrt{\sigma_{pp}}\}$, both $p \times p$ diagonal matrices. For p portfolios, we wish to test the hypothesis H: $\tau_1 = \tau_2 = \cdots = \tau_p$. Wald's statistic for testing H is given by

$$W_{Sh} = n\tilde{\boldsymbol{\tau}}^\top \boldsymbol{A}_1^\top (\boldsymbol{A}_1 \tilde{\boldsymbol{\Psi}} \boldsymbol{A}_1^\top)^{-1} \boldsymbol{A}_1 \tilde{\boldsymbol{\tau}},$$

where

$$\tilde{\boldsymbol{\Psi}} = \tilde{\boldsymbol{D}}_1 \boldsymbol{S} \tilde{\boldsymbol{D}}_1^\top + \tilde{\boldsymbol{D}}_2\{2(1+\kappa)\boldsymbol{S} \oplus \boldsymbol{S} + \kappa \boldsymbol{V}\boldsymbol{V}^\top\}\tilde{\boldsymbol{D}}_2^\top$$

is a consistent estimator of $\boldsymbol{\Psi}$, with

$$\tilde{\boldsymbol{D}}_1 = \mathrm{diag}\left\{\frac{1}{\sqrt{s_{11}}}, \ldots, \frac{1}{\sqrt{s_{pp}}}\right\} \quad \text{and} \quad \tilde{\boldsymbol{D}}_2 = -\frac{1}{2}\ \mathrm{diag}\left\{\frac{\tilde{\tau}_1}{\sqrt{s_{11}}}, \ldots, \frac{\tilde{\tau}_1}{\sqrt{s_{pp}}}\right\}.$$

The statistic W_{Sh} generalizes results from Jobson and Korkie (1981) and Memmel (2003), who consider a test for the difference between Sharpe ratios under the assumption of multivariate normality; see also Leung and Wong (2008).

5. Conclusions

In this paper, we have discussed some tests, based on large samples, to test hypotheses about means and variances of correlated random variables. Much of the literature discusses this type of tests under the assumption of normality. We have extended these tests to the class of symmetric multivariate distributions with finite fourth moments. The implementation of the tests is very simple and they can be used to compare measurement instruments. We have also extended a test to evaluate the performance of investment portfolios. The behavior of the tests discussed here for the case of finite samples, via simulation studies, and the extension of these tests to multivariate skew-elliptic distributions are topics of our current work.

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